The Value of Performance Signals Under Limited Liability

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Keywords: informativeness principle, limited liability, option repricing, pay-for-luck, performance-based vesting, performance-sensitive debt

JEL Classifications: D86, G32, G34, J33

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Abstract

This paper studies the value of additional performance signals under limited liability. We show that – contrary to the informativeness principle – informative signals may have no value, because the payment cannot be adjusted to reflect the signal realization. We derive new conditions for a signal to have value under limited liability, and study how valuable signals should be incorporated into the contract. In a compensation setting, we show precisely how the signal realization should change the number of vesting options and the option strike price, providing guidance for performance-based vesting. Surprisingly, it may be optimal for more options to vest upon a negative signal of effort. In a financing setting, our results also have implications for whether the debt repayment should be performance sensitive.

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Executive compensation contracts are typically based on multiple signals of performance. For example, Bettis et al. (2018) find that, in 2012, 70% of large U.S. firms paid their executives with performance-vesting equity, where the number of securities granted depends on performance relative to a threshold (or set of thresholds). 86% of such grants employ at least one accounting threshold, and so their value depends on factors other than the stock price – the standard "output" measure for executive contracts. Murphy's (2013) survey reports that companies use a variety of financial and non-financial performance measures when determining CEO bonuses. Additional performance signals are also used in financing contracts. Manso, Strulovici, and Tchistyi (2010) document that 40% of loans have performance pricing provisions, where the coupon rate depends on signals such as the firm's credit rating, leverage, and solvency ratios. Thus, the payment to investors depends on factors other than cash flow – the standard "output" measure for financing contracts.

The main theoretical justification for including additional performance measures is Holmström's (1979) informativeness principle. This principle states that any signal should be included in a contract if it provides incremental information about the agent's performance, over and above the information already conveyed in output (the sufficient statistic result). However, real-life contracts appear to violate the principle. Even though some contracts are based on signals other than output, many are not. Most debt does not have performance pricing provisions, and some executive stock and options do not exhibit performance-based vesting. Are these violations efficient? When should contracts depend on additional performance signals, which signals should be used, and how should they be incorporated into the contract? These questions are the focus of this paper.

The informativeness principle was derived assuming no contracting constraints. However, in almost all real-life contracting settings, the agent is protected by limited liability. Limited liability of equity applies to contracts between entrepreneurs and investors; the wage paid by a firm to a worker cannot be negative. Thus, to apply the informativeness principle to many real-life settings, we must first study whether the principle holds under limited liability, and if necessary extend it.[1]

The first contribution of this paper is to derive necessary and sufficient conditions under which contracts should be based not only on output q, but also an additional performance signal s, under limited liability. For example, q may be the stock price and s may be accounting profits.

[1] Indeed, Holmström (1979) conjectures that "If, for administrative reasons, one has restricted attention a priori to a limited class of contracts ... informativeness may not be sufficient for improvements within this class." We formally analyze the circumstances in which informative signals have value, under agent limited liability.

In this setting, the principal's problem is whether to make the manager's pay dependent purely upon the stock price, as with traditional equity grants, or also upon profits, via performance-vesting equity or a profit-contingent bonus (e.g. Li and Wang (2016)). Alternatively, s may be a stock price index of peer firms, in which case the problem is whether to engage in relative performance evaluation, or a non-accounting measure such as workplace safety.

We first study the standard framework of risk neutrality and limited liability on the manager, originally analyzed by Innes (1990). As in Innes (1990), we include a monotonicity constraint which requires the principal's payoff to be non-decreasing in output.The optimal contract is an option on output with strike price q^*. The only non-trivial dimension of the contract is the strike price q^*: the optimal contract always involves a zero payment below the strike price and the residual above it. Thus, an additional signal will only be included if the firm wishes to use its realization to vary the strike price – it will not use it to change any other dimension of the contract. If the signal suggests the manager has worked (shirked), the firm generally decreases (increases) the strike price.

In the original informativeness principle, what matters is whether a signal is incrementally informative about effort – provides incremental information about effort over and above the output realization – at any output level (in which case it has strictly positive value) or at no output level (in which case it has zero value). Under contracting constraints, we show that whether a signal has value depends on whether it is informative about effort at a specific output level, q^*. A signal that is only informative about effort for $q < q^*$ has no value for the contract. Even if the signal indicated that the manager has shirked (i.e. low q is due to low effort rather than bad luck), the principal could not use the signal to reduce the payment since the manager is receiving zero anyway: the limited liability constraint binds. Likewise, for $q > q^*$, a signal that suggests high effort has no value: the principal could not use the signal to increase the payment since the monotonicity constraint binds. (While the monotonicity constraint leads to realistic contracts, it is not necessary for the result that an informative signal may have zero value.)[2] In sum, a signal that redistributes probability mass either to the left or the right of q^* is of no value. It only has value if it leads to the principal optimally changing the strike price q^* with the signal realization. Thus, a signal can be informative almost everywhere yet still have zero value. We illustrate this point with a number of real-life examples where informative signals may not be incorporated in the contract.

[2] If monotonicity is replaced by a limited liability constraint on the principal, Innes (1990) shows that the optimal contract is "live-or-die" – the manager receives zero if output is below a threshold and the full output (rather than the residual) otherwise. Above this threshold, the contract is bounded by the limited liability constraint on the manager, rather than the monotonicity constraint.

We then extend the model to risk aversion. The contract takes a more general form – while it remains the case that the manager is paid zero below a threshold and a strictly positive amount above it, the payment above the threshold is typically non-linear (unlike with an option contact). However, it remains the case that informative signals have strictly positive value only if they are informative at output levels where limited liability does not bind, rather than at any output level as in the original informativeness principle.

The generally stronger conditions for a signal to have value under contracting constraints may explain why real-life contracts do not depend on as many signals as the original informativeness principle suggests they should, i.e. are simpler than implied by the principle.[3] For example, executive contracts typically do not depend on the firm's recovery rate in bankruptcy or the outcome of litigation against the firm, because bankruptcy and litigation typically lead to the manager being fired anyway and so he cannot be punished further. Relatedly, pay-for-luck need not be inefficient if it applies to firing decisions as found by Jenter and Kanaan (2015). On the other hand, our model does suggest that pay-for-luck is suboptimal at moderate output realizations. Indeed, we do not argue that real-life contracts are efficient. Rather, before concluding that they must be suboptimal because they violate the original informativeness principle, one must first extend the informativeness principle to take into account contracting constraints and only then make an assessment.

The second contribution of this paper is to study how to incorporate valuable signals into a contract. Doing so requires a clear characterization of the contract. We start by deriving the first set of sufficient conditions for options to be the optimal contract when the agent is risk-averse – log utility, normally-distributed output, limited liability on the manager, and a sufficiently convex cost of effort. (Innes (1990) derives conditions for options to be optimal under risk neutrality.) We also establish a sufficient condition for the validity of the first-order approach in this setting. Unlike in the risk-neutral model where the manager is the residual claimant above the threshold, under risk aversion the sensitivity of the contract above the threshold – which represents the number of options granted – is endogenously adjusted to balance the trade-off between incentives and risk-sharing. The model thus allows us to study how signal realizations should affect the number of options granted ex-post, as is the case for performance-based vesting. Despite its popularity, we are unaware of any theories that study under what conditions performance-based vesting is optimal, and what performance signals

[3] Salanié (1997, p128-129) writes that "the sufficient statistic theorem indicates that the optimal wage schedule should depend on all signals that may bring information on the action chosen by the agent. ... This prediction does not accord well with experience; real-life contracts appear ... to depend on a small number of variables only."

should be used.

Simple intuition may suggest that the number of options should depend on a signal if it provides incremental information over and above that contained in output. Our framework provides a tractable decomposition of the channels through which a signal provides information. One channel is indeed the above intuition – the signal is individually informative about effort (the "individual informativeness effect"). However, our decomposition also shows that a signal can provide information through affecting the information output provides about effort, either by shifting the output distribution (the "location effect"), or by affecting its informativeness (the "precision effect").

We show that the effect of a signal on vesting depends only on the precision effect. This effect arises because a signal realization, such as good economic conditions, is effectively a "state of nature", and output may be a more precise measure of effort – the signal-to-noise ratio may be higher – in some states than others. For example, more options should vest in good economic conditions if the manager's effort has a stronger effect on output (increasing the numerator of the signal-to-noise ratio) or if the volatility of output is lower (decreasing its denominator). In contrast, a simple application of relative performance evaluation would suggest that fewer options should vest in good economic conditions. The above result also implies that an individually informative signal will not affect vesting if it does not affect the precision of output as an effort measure, and an individually uninformative signal will affect vesting if it does. For example, economic conditions are outside the manager's control and thus individually uninformative, but should still affect vesting.

Our framework can also be used to understand how option strike prices should vary with performance signals. This depends, in part, on the individual informativeness effect as intuition might suggest.[4] Option repricing (which, empirically, nearly always involves a lowering of the strike price) can thus be justified if prompted by positive signals of CEO effort, such as a high employee satisfaction score. However, the impact of the signal on the strike price also depends on the location effect: how the signal shifts the output distribution. For example, all output levels are a more positive signal of effort if peer performance were weak than if it were strong which, ceteris paribus, will lead to a lower strike price.

The location effect means that it may sometimes be optimal to lower the strike price upon a signal that individually conveys bad news about CEO effort, contrary to conventional wisdom that such repricing necessarily results from rent extraction. For example, let the signal be a credit rating, and consider a firm with high output and a low credit rating. The low credit rating individually indicates low effort. However, it also makes the high output a stronger

[4] Here, the precision effect scales the individual informativeness effect.

indicator of high effort, since it is harder to achieve high output with a low credit rating and thus limited access to external finance. If this second consideration is sufficiently strong, the manager's pay will be higher, and thus the strike price lower.

In addition to compensation, the risk-neutral model can also be applied to a financing setting, in which case the optimal contract is debt with face value q^* (Innes (1990)). Our results give conditions under which the payment depends not only on output, as with a standard debt contract, but also on additional signals, as with performance-sensitive debt. This dependence arises if and only if these signals are informative about whether output exceeding the face value of debt is the outcome of high effort. For example, a credit rating may be incrementally informative about effort if output is below the face value of debt (i.e. the firm defaults), since effort affects the severity of default – but the creditor receives the full output upon default anyway and so the rating should not be included in the contract.

This paper is related to the theoretical literature on pay-for-performance, surveyed by Holmström (2017). In particular, Gjesdal (1982), Amershi and Hughes (1989), Kim (1995), and Chaigneau, Edmans, and Gottlieb (2019) extend the original Holmström (1979) informativeness principle, but not to settings with contracting constraints. Chaigneau, Edmans, and Gottlieb (2018) study the effect on the optimal contract of increasing the precision of output, but not the introduction of additional signals and thus do not have implications for performance-sensitive debt, performance-vesting options, or option repricing. Other theories have proposed different justifications for why contracts may not depend on additional signals. Townsend (1979) and Gale and Hellwig (1985) show that, if verifying the state is costly, optimal contracts should not involve verification of – and thus be contingent upon – the state for certain realizations. Our paper shows that even freely-verifiable signals (e.g. peer performance) may optimally not be used. Allen and Gale (1992) propose that signals may not be used if they may be manipulated. A quite separate rationale is a preference for simplicity; see Gabaix (2014) for such a model in a consumer setting. In Innes (1990), the agent's wage is zero when output falls below a threshold. Even though lower outputs are associated with lower likelihood ratios, the agent's wage does not fall. In this sense, the contract does not use all the information in output due to limited liability, similar to why additional signals may not be used in our setting. Our main contribution is not only to point out that the original informativeness principle may fail under limited liability, but also to derive necessary and sufficient conditions for an additional signal – over and above output – to have value under limited liability.

Moving to the applied literature on pay-for-performance, Dittmann, Maug, and Zhang (2011) quantify the effect on pay and firm value of various restrictions on CEO pay – restrictions on ex-post payments, ex-ante expected pay, and specific components of pay. Their calibration differs from our optimal contracting approach. Dittmann, Maug, and Spalt (2013) calibrate the cost savings from incorporating peer performance in executive contracts, and Johnson and Tian (2000) compare the incentives provided by indexed and non-indexed options. Oyer (2004), Axelson and Baliga (2009), Gopalan, Milbourn, and Song (2010), Hoffmann and Pfeil (2010), and Hartman-Glaser and Hébert (2019) provide different rationalizations for pay-for-luck. These rationalizations suggest that pay-for-luck is either always optimal or always suboptimal in a given firm. We show that, within a given firm, whether pay-for-luck is optimal depends on the output realization. In particular, a signal of peer performance that is informative about effort only at low output levels will not be incorporated into the contract. Parlour and Rajan (2019) show that, even if a credit rating contains no new information, a manager's contract may optimally depend on it because it makes an otherwise uncontractible state contractible. Manso, Strulovici, and Tchistyi (2010) offers an explanation for performance-sensitive debt based on adverse selection. Ours is based on moral hazard, and so the value of a signal depends on whether it is informative about effort at output levels where contracting constraints do not bind.

1 The Model

We consider a principal (firm) and an agent (manager). The manager is protected by limited liability and has zero reservation utility. He exerts unobservable effort of $e \in \{0, 1\}$, where $e = 0$ ("low effort") costs the manager 0, and $e = 1$ ("high effort") costs $C > 0$. As is standard, effort can be interpreted as any action that improves output but is costly to the manager, such as working rather than shirking, choosing projects that generate cash flows rather than private benefits, or not extracting rents. In this section, we assume that both the manager and firm are risk-neutral; Section 2 extends the model to risk aversion and a continuum of effort levels.

Effort affects the probability distribution of output, which is distributed over an interval $q \in [0, \bar{q}]$, where $\bar{q}$ may be $+\infty$, and of an additional signal $s \in \{s_1, ..., s_S\}$.[5] Both output and the signal are contractible. We refer to an output/signal realization (q, s) as a "state" and assume that the distribution of (q, s) conditional on any e has full support.[6]

Conditional on effort e and signal s, output q is distributed according to the probability

[5] A discrete signal space ensures that an optimal contract exists in all variations of the model that we consider. Apart from existence, however, it is straightforward to extend our results to continuous signals.

[6] The results are robust to relaxing this assumption, except that the optimal contract might not be unique. There could exist other optimal contracts that differ on a set of outputs that occur with probability zero.

density function ("PDF"):

$$f\left(q|e,s\right) := \begin{cases} \pi_s\left(q\right) & \text{if } e=1 \\ p_s\left(q\right) & \text{if } e=0 \end{cases}.$$

The marginal distribution of the signal is represented by $\phi_{e'}^{s'} := \Pr\left(s=s'|e=e'\right) > 0$. Their product yields the joint distribution of (q,s) conditional on effort, which we denote $f\left(q,s|e\right)$. The marginal distribution of output is given by

$$f\left(q|e\right) = \sum_s \phi_e^s f\left(q|e,s\right). \tag{1}$$

Let

$$LR_s\left(q\right) := \frac{\phi_1^s \pi_s\left(q\right)}{\phi_0^s p_s\left(q\right)} \tag{2}$$

denote the likelihood ratio associated with output q and signal s. When the likelihood ratio depends on s, the signal is incrementally informative about effort – i.e. it provides information about effort over and above that contained in output. We assume that the output distribution satisfies the strict monotone likelihood ratio property ("MLRP"): $LR_s(q)$ is strictly increasing in q for all s. As Holmström (1979) discusses, the principal's problem resembles a hypothesis test, where she tests the null that the agent worked against the alternative that he shirked. The likelihood ratio compares the likelihood of the null to the alternative, and the problem is whether the signal s provides additional information to guide this hypothesis test (of course, in equilibrium, the principal knows that the agent worked).

The firm has full bargaining power and offers the manager a payment conditional on the state $\{w_s\left(q\right)\}$. We assume that the gain from effort $\mathbb{E}\left[q|e=1\right] - \mathbb{E}\left[q|e=0\right]$ is sufficiently higher than the cost of effort C that the firm wishes to implement high effort, else the optimal contract would trivially involve a constant payment of zero. The firm thus solves the following program:

$$\min_{\{w_s(q)\}} \sum_s \int_0^{\bar{q}} w_s\left(q\right) \phi_1^s \pi_s\left(q\right) dq \tag{3}$$

$$s.t. \sum_s \int_0^{\bar{q}} w_s\left(q\right) \phi_1^s \pi_s\left(q\right) dq - C \geq 0 \tag{4}$$

$$\sum_s \int_0^{\bar{q}} w_s\left(q\right) \left[\phi_1^s \pi_s\left(q\right) - \phi_0^s p_s\left(q\right)\right] dq \geq C \tag{5}$$

$$w_s\left(q\right) \geq 0 \quad \forall q,s. \tag{6}$$

It minimizes the expected payment (3) subject to the manager's individual rationality constraint ("IR") (4), incentive compatibility constraint ("IC") (5), and limited liability constraint ("LL") (6). IC (5) and LL (6) imply that IR (4) is automatically satisfied, and so we ignore it in the analysis that follows.

Without limited liability on the manager, the principal could implement the first best by selling the firm to him. Since the first best is achieved, any new signal automatically has zero value and so any contracting constraint must weakly increase the value of information. Thus, it is not the case that signals always have less value under contracting constraints, as intuition might suggest. We consider limited liability on the manager throughout the paper, since this constraint is relevant for both compensation and financing contracts.

Innes (1990) considers one of two additional constraints. The first is limited liability on the firm (as well as the manager). He shows that the optimal contract is "live-or-die" – the manager receives zero if output is below a threshold, and the entire output if it exceeds it. Conversely, the firm receives the entire output if it is below the threshold, and zero if it exceeds it. Since the agent's payoff is highly discontinuous and the principal's payoff is non-monotonic in output, each party has strong incentives to manipulate output. If output were just above the threshold, the principal would exercise her control rights to "burn" output, reducing it to just below the threshold and raising her payoff from zero to the entire output. If output were just below the threshold, the manager would inject his own money into the firm to increase output, raising his payoff from zero to the entire output. Indeed, "live-or-die" contracts are almost never used in reality, potentially due to the strong manipulation incentives.

The second constraint considered by Innes (1990) prevents such manipulation. It is given by the following:

$$w_s\left(q+\epsilon\right) - w_s\left(q\right) \leq \epsilon \tag{7}$$

for all $\epsilon > 0$. Constraint (7) means that a dollar increase in output cannot increase the payment to the manager by more than a dollar, or equivalently the payoff to the firm cannot decrease in output (hence, it is often referred to as a monotonicity constraint). We assume the monotonicity constraint throughout this section; as we will soon show, it leads to contracts commonly observed in reality. However, it is not necessary for our key results: that informative signals may have zero value under limited liability, and only signals that are informative at a threshold output have value. Indeed, in Appendix B, we show that these results continue to hold if we replace the monotonicity constraint with limited liability on the firm (Innes's first setting). Moreover, they continue to hold if we remove the monotonicity constraint with no replacement at all, i.e. if the only constraint is limited liability on the manager.

Let

$$\overline{LR}_s\left(\tilde{q}\right) := \frac{\phi_1^s \int_{\tilde{q}}^{\bar{q}} \pi_s(z)dz}{\phi_0^s \int_{\tilde{q}}^{\bar{q}} p_s(z)dz} = \frac{\Pr\left(q \geq \tilde{q}, s = \tilde{s} | e = 1\right)}{\Pr\left(q \geq \tilde{q}, s = \tilde{s} | e = 0\right)} \tag{8}$$

denote the likelihood ratio associated with the event $(q \geq \tilde{q}, s = \tilde{s})$, which is strictly increasing by MLRP (as shown in Appendix A). The two terms in (8) show that a signal can affect the likelihood ratio in two ways: it can either be individually informative about effort (i.e. affect $\frac{\phi_1^s}{\phi_0^s}$), or it can affect the information output provides about effort $\frac{\int_{\tilde{q}}^{\bar{q}} \pi_s(z)dz}{\int_{\tilde{q}}^{\bar{q}} p_s(z)dz}$. Even if a signal is unaffected by effort and thus not individually informative about effort, it can still affect the likelihood ratio. For example, even if effort does not affect economic conditions, these conditions may still affect the likelihood ratio since output may be less informative about effort in booms, when all firms perform well regardless of managerial effort, than in recessions.

For each fixed κ and signal realization s, construct the threshold "strike price" as follows:

$$q_s^*\left(\kappa\right) := \begin{cases} 0 & \text{if } \overline{LR}_s(0) > \kappa \\ \bar{q} & \text{if } \overline{LR}_s(\bar{q}) < \kappa \\ \overline{LR}_s^{-1}\left(\kappa\right) & \text{if } \overline{LR}_s(0) \leq \kappa \leq \overline{LR}_s(\bar{q}) \end{cases}. \tag{9}$$

The threshold for the likelihood ratio κ is chosen so that the IC binds (existence is shown in Appendix A); if more than one such threshold exists, we choose the largest one to minimize the cost of the contract:

$$\kappa := \sup\left\{\hat{\kappa} : \sum_s \int_{\overline{LR}_s(q) > \hat{\kappa}} \left(q - q_s^*(\hat{\kappa})\right)\left[\phi_1^s \pi_s\left(q\right) - \phi_0^s p_s\left(q\right)\right] dq = C\right\} \in (0, \bar{q}). \tag{10}$$

The optimal contract is given by Lemma 1 below:

Lemma 1 *The optimal contract under risk neutrality and monotonicity is* $w_s(q) = \max\left\{q - q_s^*(\kappa), 0\right\}$, *where* $q_s^*(\kappa)$ *and* κ *are determined by (9) and (10).*

The optimal contract is an option, as in Innes (1990). If output exceeds the strike price q_s^* (which can depend on the signal realization s), the manager receives the residual $q - q_s^*$, and zero otherwise. The intuition is as follows. The absolute value of the likelihood ratio is highest in the tails of the distribution of q, so output is most informative about effort in the tails. The firm cannot incentivize the manager in the left tail by giving negative payments (due to limited liability), so it incentivizes him in the right tail by giving high payments. Under the monotonicity constraint, the maximum possible incentives involve the manager gaining one-for-one from any increase in output, so he receives the residual. Since options are typically written on the firm's stock price, we will sometimes refer to output q as the stock price.

The optimal strike price associated with signal realization s depends on the likelihood ratio of the event $q \geq q_s^*$. Note that the relevant likelihood ratio is over a range of outputs, rather than at a single output level. This is because changing the strike price q_s^* affects the payment at all output levels exceeding q_s^* – the firm cannot change the payment at specific output levels in isolation as this would violate the monotonicity constraint. An alternative intuition is as follows. Without contracting constraints, the payment for a given output level depends on the likelihood ratio at that output level; higher likelihood ratios are stronger indicators of effort and thus correspond to higher payments. Since the payment typically varies according to the specific output produced, the principal needs to know the exact output level in order to infer effort and determine the appropriate payment. However, with an option contract, the only information the principal needs to know to infer effort and determine the payment (i.e. whether the manager gets zero or the residual $q - q_s^*$) is whether the output level exceeded q_s^*, rather than the actual output level.[7] Thus, the relevant likelihood ratio is that associated with the event $q \geq q_s^*$, and this likelihood ratio affects the choice of q_s^*.

Proposition 1 gives a necessary and sufficient condition under which the contract is independent of the signal, i.e. $q_s^* = q^* \;\forall\, s$.

Proposition 1 *(Signal Has No Value): The optimal contract under risk neutrality and monotonicity is independent of the signal if and only if $\overline{LR}_s^{-1}(\kappa)$ does not depend on s, where κ is determined by (10).*

If $\overline{LR}_s^{-1}(\kappa)$ does not depend on s, we have $q^* = \overline{LR}_s^{-1}(\kappa)$ for any s, and we have:

$$\overline{LR}_{s_i}(q^*) = \overline{LR}_{s_j}(q^*) = \kappa \;\forall s_i, s_j. \tag{11}$$

The firm optimally sets the same strike price q^* if and only if the likelihood ratio that $q \geq q^*$ is always κ, regardless of s. With a binding IC, q^* solves the following equation:

$$\int_{q^*}^{\bar{q}} (q - q^*) \left[\pi(q) - p(q)\right] = C, \tag{12}$$

where $\pi(q) := \sum_s \pi_s(q)$ and $p(q) := \sum_s p_s(q)$. The threshold q^* is thus defined as a function of model primitives.

[7] The actual output level q automatically affects the payment $q - q_s^*$, but is not used to provide any inference about effort. Once the firm has observed that $q \geq q_s^*$, it knows that the likelihood ratio is sufficiently high for the manager to receive the residual.

Proposition 1 shows that limited liability requires us to refine the informativeness principle. A signal has positive value if and only if it affects the firm's optimal choice of the strike price q^*, since this is the only element of the contract that the firm can change according to the signal realization. It cannot change the contract for $q < q^*$ because it is already paying zero, nor for $q > q^*$ because it is already paying the residual. In turn, the strike price q^* depends on the likelihood ratio associated with $q \geq q^*$. Thus, a signal is valuable if and only if it is informative about whether $q \geq q^*$ is the outcome of high or low effort – i.e. provides incremental information about effort over and above the knowledge that output exceeded q^*. When $q_s^* = q^*$ – i.e. the firm would choose not to make the strike price depend on the signal – the signal has zero value because the firm cannot use it. Signals that are only informative at the tails, i.e., that affect the likelihood ratio only above or below q^*, have zero value. Note that the tails do not refer only to extreme outputs – any output realization other than q^* is a tail realization. Thus, a signal can be informative almost everywhere and still have zero value.

Example 1 *Consider a signal $s \in \{s_1, s_2\}$ which is individually uninformative about effort ($\frac{\phi_1^{s_1}}{\phi_0^{s_1}} = \frac{\phi_1^{s_2}}{\phi_0^{s_2}}$) but affects the degree to which tail outputs are informative about effort. Formally, let $F(\tilde{q}|e,s) := \int_0^{\tilde{q}} f(q|e,s)dq$, $0 \leq q_0 \leq q_1 \leq \bar{q}$, and:*

$$F(q|e,s_1) \begin{cases} < F(q|e,s_2) & \text{if } q < q_0, \\ = F(q|e,s_2) & \text{if } q \in [q_0, q_1], \\ > F(q|e,s_2) & \text{if } q > q_1. \end{cases}$$

The contract does not depend on the signal s if and only if:

$$\frac{1 - F(q^*|e=1,s_1)}{1 - F(q^*|e=0,s_1)} = \frac{1 - F(q^*|e=1,s_2)}{1 - F(q^*|e=0,s_2)},$$

which is true whenever $q^* \in [q_0, q_1]$. In this case, even though output is more informative about effort under s_1 than under s_2,[8] the optimal contract is identical under both signal realizations.

In sum, if output q is a sufficient statistic for effort e given (q, s), the signal s has zero value. However, even if q is not a sufficient statistic, s still has zero value if it is uninformative about whether the event that $q \geq q^*$ is the outcome of high or low effort, where q^* is determined in equation (12). While risk neutrality and limited liability is sometimes seen as an alternative to risk aversion in a contracting model (both are ways of ruling out the first-best solution of

[8] If expected output is the same under s_1 and s_2, then the distribution of output under s_2 is a mean-preserving spread of the distribution of output under s_1.

the principal selling the firm to the agent), the conditions for a signal to have value are much stronger under the former.

Our theoretical result on the conditions for a signal to have value under limited liability in turn leads to several applied implications for compensation contracts. First, they identify the settings in which boards should invest in additional signals of manager performance, for instance through monitoring. A signal only has value if it shifts probability mass from below q^* to above q^* (or vice-versa). A signal that redistributes mass within the left tail, or within the right tail, has zero value. A "smoking gun" indicates that a bad event is due to poor performance rather than bad luck, but the bad event will likely lead to firing anyway.[9] For instance, investors only noticed that Enron was adopting misleading accounting practices when it was already going bankrupt. Relatedly, the threshold output can be interpreted as a performance target below which the manager is fired. Signals are then only useful if they affect this target.

Second, our results imply that pay-for-luck (i.e. not obtaining signals to verify whether an output level was due to effort or luck) need not be suboptimal if it occurs at tail output realizations. Sometimes, pay-for-luck concerns very good or very bad outcomes – for example, Bertrand and Mullainathan (2001) consider how CEO pay varies with spikes and troughs in the oil price, and Jenter and Kanaan (2015) find that peer-group performance does not affect CEO firing decisions – but additional signals are only valuable for moderate outcomes. In turn, if constraints are more likely to bind in certain economic conditions (e.g. if limited liability is more likely to bind in a downturn), then the extent of pay-for-luck will be higher in these conditions.

Proposition 1 also has implications for debt contracts. Our model can be interpreted in two ways. First, the firm offers a compensation contract to the manager, as in the above exposition. Second, the manager is an entrepreneur who raises financing from an investor, which is the exposition in Innes (1990). The optimal contract is debt, and so a signal has no value in determining the repayment schedule, which is automatically the entire output if performance is poor, and the entire promised repayment (principal plus interest) if performance is good. The signal has value if and only if it affects the promised repayment. In theory, this amount could depend on many signals, but in practice it is often signal-independent. Proposition 1 potentially rationalizes this practice – even if signals are informative about effort, they should not enter the contract if they are only informative in the tails. In addition, Proposition 1 provides conditions under which the repayment *should* depend on additional signals, as in performance-sensitive debt, where the repayment is higher upon negative signals of borrower

[9] The "smoking gun" could be generated by an audit that is only undertaken upon a bad event, in which case the signal realization is zero absent a bad event.

performance. This is the case if and only if the signal is informative about effort conditional on output exceeding the promised repayment.

This financing application is relevant for both mature firms and also young firms since they frequently raise debt and the entrepreneur holds levered equity, as shown by Robb and Robinson (2014). Indeed, the model also allows us to study the conditions under which the entrepreneur's equity claim should depend on performance milestones, as documented empirically by Kaplan and Strömberg (2003) in their analysis of venture capital contracts.[10]

We close with three examples that apply Proposition 1 to a real-world setting. First, we consider whether contracts should depend on s, a signal of economic conditions. Economic conditions are informative about effort – for any given level of output, a high s suggests that the output was due to good economic conditions rather than effort, and so increases the likelihood that the manager has shirked. However, Proposition 1 shows that economic conditions s should only affect the contract if they affect the probability that $q > q^*$ under high versus low effort. This will fail to hold if they affect the level of output but not the probability that output exceeds q^*.[11] For example, consider a start-up which is developing a major new software; the manager's effort affects the probability that the software is adopted by the industry. If the software is adopted, $q > q^*$ (regardless of economic conditions); if it is not adopted, $q < q^*$ (again, regardless of economic conditions). Economic conditions could affect the actual level of q (both if the software is adopted and if it is not), but if they do not affect the probability that $q > q^*$, because they do not affect the likelihood that the software will be adopted, then they should not be included in the contract. As a second example, consider a firm whose production can break down due to a fault, whose probability can depend on managerial effort. If it does, then output is below q^* (regardless of economic conditions); if it does not, then $q > q^*$ (regardless of economic conditions). As in the previous example, economic conditions could affect the actual level of q (both if production breaks down and if it does not), but if they do not affect the probability that production breaks down, then they should not be included in the contract. In the first example, the signal is uninformative about the upside (developing new software); in this example it is uninformative about the downside (production breaking down).

Second, let the signal s be the average output of other firms in the industry. Suppose that

[10] While the original informativeness principle in Holmström (1979) would suggest that contracts should depend on performance milestones, it does not generally deliver debt and equity as optimal contracts. Kaplan and Strömberg (2004) find that the debt and equity contracts used in venture capital are determined primarily by agency problems, not risk-sharing considerations.

[11] It will also hold if they affect the probabilities (that $q > q^*$ under high and low effort) by the same proportion.

output correlation is countercyclical (as found by Perez-Quiros and Timmerman (2000)): firm outputs are positively correlated when the industry is in recession, but independent otherwise. If the managers' options are out-of-the-money in a recession, then firms will not use relative performance evaluation. Even though other firms' outputs are informative about effort, the manager is paid zero anyway.[12]

Third, under the financing application, consider a firm that issued debt whose face value in the absence of an additional signal is q^*. The manager's effort affects the distribution of both output and an additional signal, the firm's credit rating, which captures the probability and severity of default. If $q < q^*$, the credit rating is informative about effort since effort affects the severity of default. If $q > q^*$, the credit rating is uninformative about effort since default can only occur due to extraneous events, such as the bankruptcy of a major customer, bank, or hedging counterparty, which is outside the manager's control. Thus, the credit rating is informative about effort only conditional upon $q < q^*$, but the manager's payoff is zero anyway. Hence, it should not be part of the contract – debt is not performance-sensitive – even though output is not a sufficient statistic for effort.

2 Continuous Effort and Risk Aversion

This section generalizes the model to both risk aversion and a continuous effort decision, retaining previous assumptions unless otherwise specified. Effort is now given by $e \in [0, \bar{e}]$. Let $F(q|e,s)$ and $f(q|e,s)$ denote the cumulative distribution function ("CDF") and PDF of q conditional on e and s. We assume that, for each s, $F(\cdot|\cdot,s)$ is twice continuously differentiable with respect to q and e. We continue to assume MLRP, which here entails $\frac{d}{dq}[\frac{f_e(q|e,s)}{f(q|e,s)}] > 0$, where $f_e(q|e,s)$ denotes the first derivative of the PDF with respect to e. We assume that the marginal distribution of the signal ϕ_e^s is differentiable with respect to e.

The manager's utility of money is given by a strictly increasing, weakly concave, twice differentiable function u. He has outside wealth $\bar{W} > 0$ and reservation utility $\overline{u}$.[13] His cost of effort $C(e)$ is a twice continuously differentiable, strictly increasing, and strictly convex function. Thus, given a contract $w_s(q)$ and an effort level e, his objective function is $\mathbb{E}[u(\bar{W} +$

[12] This contrasts Theorem 7 in Holmström (1982), which yields relative performance evaluation under weaker conditions in the absence of contracting constraints: a contract should depend exclusively on the firm's own output if and only if firm outputs are independent.

[13] With risk neutrality (Section 1) we assumed zero reservation utility, so that solving the incentive problem is costly to the principal as it involves paying the agent rents (i.e. a slack IR). With risk aversion, solving the incentive problem is costly for the principal even if the agent does not receive rents (i.e. the IR binds), since the principal must pay a premium for the risk the agent bears from receiving incentive compensation.

$w_s(q))|e] - C(e)$.

We follow Grossman and Hart (1983) and separate the principal's problem into two stages. The first stage determines the cost of implementing each effort level. Given these costs, the second stage determines which effort level to implement. We study whether the optimal contract for implementing each given effort level does not depend on the signal.[14] To implement a given effort level $\hat{e}$, the firm chooses a function $w_s(\cdot)$, for each possible value of the signal s, to solve the following problem:

$$\min_{\{w_s(q)\}f} \sum_s \phi_{\hat{e}}^s \int_0^{\overline{q}} w_s(q) f(q|\hat{e}, s) dq \tag{13}$$

subject to

$$\sum_s \phi_{\hat{e}}^s \int_0^{\overline{q}} u\left(\bar{W} + w_s(q)\right) f(q|\hat{e}, s) dq - C(\hat{e}) \geq \overline{u}, \tag{14}$$

$$\hat{e} \in \arg\max_e \sum_s \phi_e^s \int_0^{\overline{q}} u\left(\bar{W} + w_s(q)\right) f(q|e, s) dq - C(e), \tag{15}$$

$$w_s(q) \geq 0 \;\; \forall q, s. \tag{16}$$

Importantly, unlike in the risk-neutral model of Section 1, we do not need to impose a monotonicity constraint to rule out discontinuities that may induce manipulation. Intuitively, when the manager is risk-averse, the principal might not offer a discontinuous contract as it leads to inefficient risk-sharing. Thus, limited liability on the manager is the only contracting constraint that we consider.

Following Holmström (1979), Shavell (1979) and the subsequent literature on the informativeness principle (e.g. Gjesdal (1982), Kim (1995)), we assume that the first-order approach ("FOA") is valid; see Chaigneau, Edmans, and Gottlieb (2019) for the informativeness principle without the FOA. We can thus replace the IC in (15) by the following equation:

$$\sum_s \left[\frac{d\phi_{\hat{e}}^s}{de} \int_0^{\overline{q}} u\left(\bar{W} + w_s(q)\right) f(q|\hat{e}, s) dq + \phi_{\hat{e}}^s \int_0^{\overline{q}} u\left(\bar{W} + w_s(q)\right) f_e(q|\hat{e}, s) dq \right] = C'(\hat{e}) \tag{17}$$

Let λ and μ denote the nonnegative Lagrange multipliers associated with the IR (14) and IC (15), respectively. Our analysis allows for $\lambda = 0$, i.e. the case in which the IR is non-binding. The optimal contract is given by Lemma 2 below.

[14] If those conditions hold for the effort level that is most profitable for the principal, the optimal contract (with effort chosen optimally) will also not depend on the signal. A sufficient but unnecessary condition is that the conditions we identify hold for all effort levels.

Lemma 2 *The optimal contract under risk aversion satisfies:*

$$w_s(q) = \max\left\{ u'^{-1}\left(1 \Big/ \left(\lambda + \mu \left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right] \right) \right) - \bar{W}, 0 \right\}. \tag{18}$$

The contract involves a minimum payment of zero, and higher payments in states associated with high likelihood ratios. We now analyze the conditions under which the optimal contract is independent of the signal. Without the signal s, the likelihood ratio at a given value of q can be written as $LR(q) := \frac{f_e(q|\hat{e})}{f(q|\hat{e})}$. With the signal s, we define the likelihood ratio as

$$LR_s(q) := \frac{f_e(q,s|\hat{e})}{f(q,s|\hat{e})} = \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}. \tag{19}$$

As in the previous section, a signal can affect the likelihood ratio in two ways. First, it can be individually informative about effort, i.e., $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}$ depends on s. Second, it can affect the information output provides about effort, i.e., $\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}$ depends on s.

Proposition 2 gives conditions under which the optimal contract is independent of the signal. For each fixed κ and signal realization s, construct the threshold above which the payment is strictly positive as follows:

$$q_s^{**}(\kappa) := \begin{cases} 0 & \text{if } LR_s(0) > \kappa \\ q & \text{if } LR_s(\bar{q}) < \kappa \\ LR_s^{-1}(\kappa) & \text{if } LR_s(0) \leq \kappa \leq LR_s(\bar{q}) \end{cases} \tag{20}$$

The threshold likelihood ratio κ is chosen so that the IC binds for effort $\hat{e}$; if more than one such threshold exists, we choose the largest one:

$$\begin{aligned} \kappa := \quad & \sup\left\{ \hat{\kappa} : \sum_s \left[\int_{LR_s(q)\leq\hat{\kappa}} u\left(\bar{W}\right) \left[\frac{d\phi_{\hat{e}}^s}{de} f(q|\hat{e},s) + \phi_{\hat{e}}^s f_e(q|\hat{e},s) \right] dq \right.\right. \\ & \left.\left. + \int_{LR_s(q)>\hat{\kappa}} u\left(\bar{W} + w_s(q)\right) \left[\frac{d\phi_{\hat{e}}^s}{de} f(q|\hat{e},s) + \phi_{\hat{e}}^s f_e(q|\hat{e},s) \right] dq \right] = C'(\hat{e}) \right\}, \end{aligned} \tag{21}$$

where $w_s(q)$ is given by Lemma 2. It follows from equation (18) that the manager receives zero below a threshold $q_s^{**} \geq 0$ and a positive payment above this threshold. Thus, limited liability can only be non-binding in an interval of high outputs, and so only signals informative conditional on a high output can have value. Proposition 2 summarizes this reasoning.

Proposition 2 *(Signal Has No Value, Risk Aversion): The optimal contract is independent* *of the signal if and only if* $LR_{s_i}(q) = LR_{s_j}(q)\ \forall q \in [q^{**}_{s_i}, \bar{q}] \cup [q^{**}_{s_j}, \bar{q}], s_i, s_j$*, where* q^{**}_s *is given by equation (20).*

Proposition 2 states that a signal only has value if it affects the likelihood ratio at output realizations where contracting constraints do not bind. This is the same principle as in the risk-neutral model, but the set of output realizations where constraints do not bind is larger. With risk neutrality, the contract provides the maximum required incentives without violating monotonicity. Since either monotonicity or limited liability binds almost everywhere, there is a single intermediate output level q^*_s at which constraints do not bind. With risk aversion, the contract does not provide maximum incentives but trades off incentives with risk-sharing, and so constraints bind at fewer output levels. Constraints do not bind at a range of output levels $[q^{**}_s, \bar{q}]$, and so the conditions for a signal to have value are weaker.[15] However, the conditions for a signal to have value are stronger than in the original informativeness principle, where there was no limited liability.

2.1 Option Repricing and Performance-Vesting

Thus far, we have studied *whether* informative signals have value under limited liability. We now turn to a second question – *how* informative signals should be incorporated into the contract when they do have value. In the risk-averse model of this section, in general the contract will be highly complex with no closed-form solution, making it difficult to give a clear characterization of how signals should be incorporated into the contract. However, Lemma 3 shows that, under the standard assumptions of log utility, normally-distributed output and limited liability, the optimal contract is an option.[16]

Formally, conditional on the signal s, output q is normally distributed with mean $h_s(e)$ and standard deviation σ_s, where $h_s(0) = 0$ and $h'_s(e) > 0$. The manager has log utility ($u(w) = \ln w$) and limited liability. The Supplementary Appendix provides a sufficient condition for the validity of the FOA under these assumptions. This new condition is important, since the conditions proposed by Rogerson (1985) and Sinclair-Desgagné (1994) cannot be applied to

[15] With risk neutrality, a signal has value if it provides information over whether output exceeding q^*_s, i.e. lying anywhere in the interval $[q^*_s, \bar{q}]$, results from high or low effort. With risk aversion, a signal has value if it is informative about whether output equaling a specific value in $q \in \bigcup_s [q^{**}_s, \bar{q}]$ results from high or low effort.

[16] We can also analyze how an informative signal can be incorporated into the contract in the risk-neutral model of Section 1, since the contract can also be characterized. In the risk-averse model, the signal can be incorporated into the contract by changing either the number of options granted or the strike price; in the risk-neutral model it can only affect the strike price (as the manager is the residual claimant). We thus only analyze this question for the risk-averse model as we have richer results; in addition, the effect on the strike price in the risk-neutral model is similar to here.

many commonly used probability distributions including the normal distribution, while other conditions, such as those proposed by Jewitt (1988) and Jung and Kim (2015), are designed for settings without constraints on contracting. Intuitively, our condition requires the cost of effort to be sufficiently convex.[17] From (19), the likelihood ratio in this setting is given by

$$\begin{aligned} LR_s(q) := \frac{f_e(q,s|\hat{e})}{f(q,s|\hat{e})} &= \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \\ &= \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{h_s'(\hat{e})}{\sigma_s^2}\left[q - h_s(\hat{e})\right]. \end{aligned} \tag{22}$$

The $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}$ term, the "individual informativeness effect", is standard. Under the normal distribution (or any distribution with a linear likelihood ratio), the information output provides about effort $(\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)})$ can be decomposed into two terms, providing additional intuition. The first term, $\frac{h_s'(\hat{e})}{\sigma_s^2}$, is the "precision effect." The signal s increases the precision of output as a measure of effort, through increasing the impact of effort on output $h_s'(\hat{e})$ or reducing the volatility of output σ_s^2. The second term, $h_s(\hat{e})$, is the "location effect." The signal affects expected output $h_s(\hat{e})$ and thus changes the location of the output distribution. For example, in good economic conditions, the output distribution shifts to the right. A given output level is thus a less positive signal of output, and so should lead to a lower payment – the intuition behind relative performance evaluation. The optimal contract is given by Lemma 3 below.

Lemma 3 *Under the new condition for the FOA derived in the Supplementary Appendix, the optimal contract under log utility and normally-distributed output consists of $n_s^* \geq 0$ options with a strike price of q_s^{**}:*

$$w(q) = n_s^* \max\{q - q_s^{**}, 0\}, \tag{23}$$

The optimal contract gives the manager n_s^* options with strike price q_s^{**}. The intuition is as follows. Given limited liability, the minimum payment is zero; given MLRP, this minimum payment will be made for all outputs below a threshold. Above the threshold, the payment is positive and determined so that the manager's marginal utility is the inverse of a linear transformation of the likelihood ratio (see Lemma 2). With log utility, marginal utility is the inverse of the payment, and so the payment equals a linear transformation of the likelihood ratio. With normally-distributed output, the likelihood ratio is linear in output, and so the

[17] The condition for the validity of the FOA in the case without an additional signal is remarkably simple: it is $C''(e) \geq \frac{\bar{e}}{\sigma^2}$ for all $e \in [0, \bar{e}]$. For example, with a quadratic effort cost, $C(e) = \alpha e + \frac{\beta}{2}e^2$, and the condition is $\beta \geq \frac{\bar{e}}{\sigma^2}$.

payment is linear in output. Overall, the payment is zero below a threshold and linear in output above the threshold, which corresponds to an option.

To our knowledge, Lemma 3 and our condition for the validity of the FOA in this setting provide the first sufficient conditions for the optimality of options with a risk-averse manager.[18] More generally, the linearity of the contract above the threshold, and thus the optimality of the option contract, holds not only for the normal distribution but for any distribution that has a linear likelihood ratio (for example, the gamma distribution).

Having derived the optimal contract in closed form, we can now study how the signal affects each dimension of the contract. Intuitively, the number of vesting options represents the sensitivity of pay to output, whereas the strike price affects the level of pay for all output levels above it. Thus, a signal realization should be associated with more vesting options if it is associated with a higher optimal sensitivity of pay to output, due to output being a more precise measure of effort. A signal realization should be associated with a lower strike price if it increases the optimal level of pay, by indicating high effort regardless of output. Proposition 3 formally demonstrates these results.

Proposition 3 *(Effect of Signal on Vesting and Strike Price):*

(i) The number of options received ex-post by the manager n_s^* *is independent of the signal if and only if* $\frac{h_s'(\hat{e})}{\sigma_s^2}$ *does not depend on* s*.*

(ii) The strike price q_s^{**} *is such that:*

$$q_s^{**} = h_s(\hat{e}) + \frac{\sigma_s^2}{h_s'(\hat{e})}\left(K - \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}\right), \tag{24}$$

where $K \in \mathbb{R}$*. It is independent of the signal if and only if the output* q *that solves* $\frac{f_e(q,s|\hat{e})}{f(q,s|\hat{e})} = K$ *does not depend on* s*.*

The intuition for the number of options is as follows. As in any principal-agent model, pay

[18] Jewitt, Kadan, and Swinkels (2008) show that the contract is "option-like" with risk aversion and agent limited liability, in that incentives are zero for low output and positive for high output, but do not identify conditions under which the increasing portion of the contract is linear. Hemmer, Kim, and Verrecchia (1999) identify a linear likelihood ratio and log utility as leading to the contract having a linear portion, but did not combine them with limited liability to obtain an option contract. In addition, Jewitt, Kadan, and Swinkels (2008) assume the Rogerson (1985) conditions to guarantee the validity of the FOA, but they do not hold under the normal distribution; Hemmer, Kim, and Verrecchia (1999) assume the Jewitt (1988) conditions but they do not hold under limited liability. We derive a general condition for the validity of the FOA that holds in the setting of limited liability, normal output and log utility. The standard model justifying options under moral hazard is Innes (1990), which requires the agent to be risk-neutral. Unlike in the risk-neutral model where the manager is the residual claimant for $q \geq q^*$, so that the number of options is fixed at 1, under risk aversion it need not be.

is increasing in the likelihood ratio. The number of options represents the sensitivity of pay to output, and is thus increasing in the sensitivity of the likelihood ratio to output, $\frac{dLR_s(q)}{dq} = \frac{h_s'(\hat{e})}{\sigma_s^2}$. Thus, the number of options is increasing in $\frac{h_s'(\hat{e})}{\sigma_s^2}$, and so is given by $n_s^* = \mu \frac{h_s'(\hat{e})}{\sigma_s^2}$. The strike price is the level of output at which the likelihood ratio is K, i.e. the output level that, if not reached, it is sufficiently likely that the agent shirked that it is optimal to pay him zero.

2.1.1 Performance-based vesting

Part (i) of Proposition 3 studies how a signal realization affects the number of options given to the manager. Proposition 2 showed that a signal has value if it affects any component of the likelihood ratio (22) where limited liability does not bind: $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}$ (the individual informativeness effect), $\frac{h_s'(\hat{e})}{\sigma_s^2}$ (the precision effect), or $h_s(\hat{e})$ (the location effect). The existence of such a signal will, in general, alter the Lagrange multiplier μ and thus scale up or down the number of options $n_s^* = \mu \frac{h_s'(\hat{e})}{\sigma_s^2}$ received across all signal realizations. However, the number of options received will depend on the actual signal realization only if it affects $\frac{h_s'(\hat{e})}{\sigma_s^2}$ rather than $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}$ or $h_s(\hat{e})$. What matters is whether the signal realization affects the precision of output as an effort measure; it does not matter whether it is individually informative about effort or affects the location of the output distribution.

The intuition is as follows. Pay should be more sensitive to output under a particular signal realization, i.e. the number of vesting options should be greater, if output is a more precise measure of effort under this signal. This arises if either effort has a greater effect on output under this signal ($h_s'(\hat{e})$ is higher) or output is less volatile under this signal (σ_s^2 is lower). To our knowledge, this result is the first theoretical justification of why performance-based vesting may be optimal. One might think that a signal that is individually positively informative about effort (i.e. increases $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}$) should lead to more vesting, and indeed current performance-vesting practices award more equity after beating performance thresholds. However, Proposition 3 shows that positive signals of effort should increase the level of pay for all output realizations (reduce the strike price) rather than the sensitivity of pay to output (increase the number of vesting options). Similarly, one might think that a signal that indicates that high output is due to luck (i.e. the output distribution has shifted to the right) should lead to less vesting, and indeed current performance-vesting practices typically benchmark performance measures against peers. However, Proposition 3 shows that the location of the output distribution $h_s(\hat{e})$ affects the strike price, not the number of vesting options.

A signal realization is effectively a contractible state of nature. Even if the manager's effort does not affect the state, i.e. the signal is individually uninformative about effort, the number of vesting options should still depend on the state if the precision of output as an effort measure varies across states. The relative performance evaluation effect already shows that individually uninformative signals should affect the contract if they affect the location of the output distribution, i.e., good output is easier to achieve in some states. Here, we show that such signals affect the number of vesting options even if there is no location effect.

We consider two examples to apply the results of part (i) of Proposition 3. First, let s be a signal of economic conditions, which are outside the manager's control and thus individually uninformative about effort ($\frac{d\phi^s_{\hat{e}}/de}{\phi^s_{\hat{e}}}$ is independent of s). Still, they may affect vesting if they affect either $h'_s(\hat{e})$ or σ^2_s. Starting with the former ($h'_s(\hat{e})$), if good economic conditions increase the effect of the manager's effort on output $h'_s(\hat{e})$, e.g. if the effect is multiplicative in firm value, vesting should be increasing in economic conditions. This result suggests that it may be efficient for more options to vest upon *low* signals of effort – for a given output level, good economic conditions are typically a negative signal of effort, because they suggest that the output was due to good luck rather than effort. However, as discussed, this consideration will only affect the strike price. In contrast, if bad economic conditions increase the impact of effort, e.g. if industries are transformed in bad times, vesting should be decreasing in economic conditions. Moving to the latter (σ^2_s), if idiosyncratic risk[19] σ_s is lower (higher) in good economic conditions, then output is a more (less) precise signal of effort and so vesting should be higher (lower). The dependence of either $h'_s(\hat{e})$ or σ^2_s on economic conditions s shows that it may be optimal for vesting to be affected by luck. This contrasts with current performance-vesting practices which assume that vesting should depend on performance measures within the manager's control.

Second, let s be an accounting performance measure, such as profits or cash flows, which Bettis et al. (2018) show to be commonly-used vesting conditions. Unlike economic conditions, accounting performance is individually informative about effort. However, the number of vesting options depends only on the precision effect $\frac{h'_s(\hat{e})}{\sigma^2_s}$ and not the individual informativeness effect $\frac{d\phi^s_{\hat{e}}/de}{\phi^s_{\hat{e}}}$. In particular, vesting may be higher upon low profits. While low profits are individually a negative signal about effort, this consideration will increase the strike price (as we will discuss in Section 2.1.2) rather than affect vesting. If volatility σ_s is increasing in s, vesting is decreasing in profits. This may be the case for a start-up, where the baseline scenario is low profits and a low stock price. High profits increase the variability of the stock

[19]In this application, we refer to σ_s as idiosyncratic risk as it represents volatility conditional on economic conditions. Note that the unconditional variance of output will typically also depend on the variance of economic conditions. However, since economic conditions are contractible, they can be filtered out of the output measure, so that output informativeness only depends on idiosyncratic risk σ_s.

price (e.g. because investors speculate as to whether the high profits are sustainable), which makes the stock price less informative about effort. In contrast, if σ_s is decreasing in s, vesting is increasing in profits. This may be the case for a mature firm, where the baseline scenario is high profits and a high stock price. High profits imply "business as usual", where the stock price is less volatile and thus informative about effort. Low profits likely mean that the business was disrupted, for example by new entrants, so the stock price is more volatile and less informative about effort. As in the first example, s could also affect vesting by affecting the manager's effect on output $h'_s(\hat{e})$. Continuing with the case of the mature firm, if high profits loosen financial constraints and thus increase the effect of effort on output, then vesting will be increasing in firm profits.

2.1.2 Strike price

Part (ii) of Proposition 3 turns to the second dimension of the contract, the strike price. The strike price affects the level of pay for all outputs above the strike price, and so it depends on what a signal individually conveys about effort (regardless of the output realization), in addition to how the signal affects the information output provides about effort. Contrary to intuition, it may be optimal for the strike price to be lowered upon a signal that individually indicates low effort. Consider two signal realizations, L and H, such that $\frac{d\phi_{\hat{e}}^L/de}{\phi_{\hat{e}}^L} < \frac{d\phi_{\hat{e}}^H/de}{\phi_{\hat{e}}^H}$: L is individually worse news about effort than H. Despite this, the strike price may be lower under L ($q_L^{**} < q_H^{**}$). This may occur in two cases. Letting the information output provides about effort be $\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} := a_s + b_s q$, we have $a_s = -\frac{h'_s(\hat{e})}{\sigma_s^2} h_s(\hat{e})$ (the location effect scaled by the precision effect) and $b_s = \frac{h'_s(\hat{e})}{\sigma_s^2}$ (the precision effect).

One case is $a_L > a_H$ and $b_L = b_H$, so the signal has a location effect but not a precision effect. Here, any given output q is better news about effort under L than H, since $\frac{f_e(q|\hat{e},L)}{f(q|\hat{e},L)} > \frac{f_e(q|\hat{e},H)}{f(q|\hat{e},H)}$ for all q. Equation (24) shows that the strike price is optimally lower under L if the location effect (the difference between a_L and a_H) outweighs the individual informativeness effect (the difference between $\frac{d\phi_{\hat{e}}^L/de}{\phi_{\hat{e}}^L}$ and $\frac{d\phi_{\hat{e}}^H/de}{\phi_{\hat{e}}^H}$). For example, let q be the profits of an industry incumbent and s be the number of new entrants into its industry. A low number of new entrants ($s = H$) is individually a better signal of effort than a high number ($s = L$), because it is harder to enter an industry where incumbents offer good products. This consideration may be outweighed by a second effect – achieving a given level of profits in an industry with more competitors is a positive signal about effort, because the number of competitors shifts the location of the output distribution. Thus, a given level of profits should be rewarded more when there are more entrants, i.e. $q_L^{**} < q_H^{**}$. Even though many new entrants indicate low effort, a given level of profits is a stronger signal of effort if combined with more entrants, and so more entrants are associated with a lower strike price.

A second case is $b_D > b_L$, $a_D = a_L$, and $\frac{d\phi_{\hat{e}}^D/de}{\phi_{\hat{e}}^D} < \frac{d\phi_{\hat{e}}^L/de}{\phi_{\hat{e}}^L} < 0$. Here the signal has a precision effect but not a (scaled) location effect. Both signals D ("dire") and L ("low") are individually bad news about effort, with D being worse news. Since $b_D > b_L$, output q is more informative about effort under D than L, and so the manager should be rewarded more for a high output under D. This generates a lower strike price under D, if the precision effect (the difference between b_D and b_L) is sufficiently large to outweigh the individual informativeness effect (the difference between $\frac{d\phi_{\hat{e}}^D/de}{\phi_{\hat{e}}^D}$ and $\frac{d\phi_{\hat{e}}^L/de}{\phi_{\hat{e}}^L}$). For example, consider a firm whose credit rating can be downgraded by one notch but remain investment-grade ($s = L$), or downgraded to junk ($s = D$). A downgrade to junk is individually worse news about effort than a one notch downgrade. Such a downgrade also restricts the firm's access to external financing; since it is now financially constrained, its performance may depend more on managerial effort (e.g. to cut costs or reallocate capital across divisions). Thus, output is more informative about effort. As a result, high output following a downgrade to junk can indicate effort more than high output following a one notch downgrade. Even though a downgrade to junk status individually indicates low effort, in combination with high output it indicates high effort, and so can be associated with a lower strike price ($q_D^{**} < q_L^{**}$). Since output is more informative about effort, the number of vesting options also increases. However, rewarding high output following a downgrade to junk requires reducing the strike price, rather than only increasing the number of options, as the latter will have no effect on pay if the options are out-of-the-money.

If the signal s does not affect the information output provides about effort ($\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}$ is independent of s), then the likelihood ratio $LR_s(q)$ only depends on the individual informativeness of the signal $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}$. Then, we obtain the intuitive result that a signal realization that is individually bad news about effort will be associated with a higher strike price. Overall, part (ii) provides conditions under which the strike price should depend on additional signals. This dependence can be implemented via option indexing or option repricing. Brenner, Sundaram, and Yermack (2000) find empirically that repricing nearly always involves a lowering of the strike price, and follows poor stock price performance (both absolute and industry-adjusted). Our model suggests that a reduction in the strike price should *generally* be prompted by positive, rather than negative, signals of effort, suggesting that such practices are suboptimal.[20] However, the above examples provides conditions under which such repricing is optimal, con-

[20] Acharya, John, and Sundaram (2000) also study the repricing of options theoretically. In their model, repricing is not undertaken to make use of additional informative signals, but instead to maintain effort incentives when options fall out of the money.

trary to concerns that it is universally inefficient because it rewards failure (e.g. Bebchuk and Fried (2004)).

3 Conclusion

This paper investigates studies the conditions under which additional signals of performance have value for a contract under agent limited liability, an important feature in virtually all real-life contracting settings. We show that the conditions for a signal to have value are much stronger than in the original informativeness principle, which was derived assuming unlimited liability. As a result, it may be optimal not to incorporate some informative signals into a contract. Under risk aversion and unlimited liability, the original informativeness principle states that a signal has value if and only if it is informative about effort at any output level. We show that, under risk aversion and limited liability, a signal has value if and only if it is informative at output levels where limited liability does not bind. If the agent is risk-neutral, the conditions are even stronger – a signal has value if and only if it is informative about whether beating a threshold output level is more likely to have resulted from high effort than low effort.

In addition to the theoretical contribution of new conditions for a signal to have value in the presence of limited liability, the results have a number of implications for real-life contracts. Starting with compensation contracts, our results offer a potential explanation as to why both pay and the firing decision do not depend on many potentially informative signals, when it is optimal and not optimal to filter out luck, when options should be repriced, and whether options should have performance-based vesting conditions. For example, performance-based vesting is not necessarily optimal even if a signal is incrementally informative about effort; instead, it must affect the precision of output as an effort measure, by affecting either the impact of effort on output or the volatility of output. Surprisingly, the strike price of an option may optimally fall, or the number of vesting options may optimally rise, upon a signal that is individually bad news about effort. Moving to financing contracts, the results suggest whether and under what conditions debt should be performance-sensitive.

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A Proofs

Proof that (8) is strictly increasing in q.

Differentiate the expression for the LR in (8):

$$\frac{d}{dq}\left\{\frac{\phi_1^s\int_q^{\bar{q}}\pi_s(z)dz}{\phi_0^s\int_q^{\bar{q}}p_s(z)dz}\right\}=\frac{\phi_1^s}{\phi_0^s}\frac{-\pi_s(q)\int_q^{\bar{q}}p_s(z)dz+p_s(q)\int_q^{\bar{q}}\pi_s(z)dz}{\left(\int_q^{\bar{q}}p_s(z)dz\right)^2},$$

which is positive if and only if

$$\frac{\pi_s(q)}{p_s(q)}<\frac{\int_q^{\bar{q}}\pi_s(z)dz}{\int_q^{\bar{q}}p_s(z)dz}\ \Leftrightarrow\ \int_q^{\bar{q}}\frac{\pi_s(z)}{\pi_s(q)}dz>\int_q^{\bar{q}}\frac{p_s(z)}{p_s(q)}dz\ \Leftrightarrow\ \int_q^{\bar{q}}\left[\frac{\pi_s(z)}{\pi_s(q)}-\frac{p_s(z)}{p_s(q)}\right]dz>0.$$

This is satisfied because, for any $z>q$, MLRP guarantees that $\frac{\pi_s(z)}{p_s(z)}>\frac{\pi_s(q)}{p_s(q)}$. ■

Proof of Lemma 1. The proof is divided into two parts:

Step 1. Conditional on each signal realization, the optimal contract is an option.

This part adapts the argument from Matthews (2001) to show that the optimal contract gives the manager an option with payoff $\max\{q-q_s,0\}$ for some strike price q_s. Let $w_s(q)$ be a contract satisfying the LL, monotonicity, and the IC. Notice that there exists a unique option contract with the same expected payment conditional on each signal realization. In other words, for each s, there exists a unique q_s that solves

$$\int_0^{\bar{q}}\max\{q-q_s,0\}\pi_s(q)dq=\int_0^{\bar{q}}w_s(q)\pi_s(q)dq. \tag{25}$$

Suppose $w_s(q)\neq\max\{q-q_s,0\}$ in a set of states with positive measure. We claim that the manager's incentives to shirk are higher under $w_s(q)$ than with the option contract:

$$\int_0^{\bar{q}}w_s(q)p_s(q)dq>\int_0^{\bar{q}}\max\{q-q_s,0\}p_s(q)dq.$$

Let $v_s(q):=w_s(q)-\max\{q-q_s,0\}$. Since $v_s(q)\neq0$ with positive probability and it has mean zero, it must be strictly positive and strictly negative in sets of states with positive probability. Moreover, because $w_s(q)$ satisfies the LL and monotonicity, there exists $k\in(0,\bar{q})$ such that

$v_s(q) \geq 0$ if $q \leq k$ and $v_s(q) \leq 0$ if $q \geq k$. Then,

$$\begin{aligned} 0 &= \int_0^{\bar{q}} v_s\left(q\right)\pi_s\left(q\right)dq \\ &= \int_0^{\bar{q}} v_s\left(q\right)\tfrac{\pi_s(q)}{p_s(q)}p_s(q)dq \\ &= \int_0^{k} v_s\left(q\right)\tfrac{\pi_s(q)}{p_s(q)}p_s(q)dq + \int_k^{\bar{q}} v_s\left(q\right)\tfrac{\pi_s(q)}{p_s(q)}p_s(q)dq \\ &< \int_0^{k} v_s\left(q\right)\tfrac{\pi_s(k)}{p_s(k)}p_s(q)dq + \int_k^{\bar{q}} v_s\left(q\right)\tfrac{\pi_s(k)}{p_s(k)}p_s(q)dq \\ &= \tfrac{\pi_s(k)}{p_s(k)}\int_0^{\bar{q}} v_s\left(q\right)p_s(q)dq, \end{aligned} \tag{26}$$

where the first line uses the fact that $v_s(q)$ has mean zero under high effort; the second multiplies and divides by $p_s(q)$, the third splits the integral between the positive and negative values of $v_s(q)$; the fourth uses MLRP and the fact that the terms in the first integral are positive whereas the ones in the second integral are negative; and the last line regroups the integrals.

Thus, conditional on each signal realization s, shirking gives the manager a higher payment with the original contract than with the option. Moreover, both contracts pay the same expected amount when the manager exerts effort. We have therefore shown that substituting a non-option contract with an option allows the firm to relax the IC. Since the IC must bind at the optimum, this establishes that the original contract cannot be optimal.

Step 2. Determining the optimal strike prices.

Since any option contract satisfies the LL and monotonicity, the firm's program becomes:

$$\min_{\{q_s\}_{s=1,\ldots,S}} \sum_s \int_{q_s}^{\bar{q}} \left(q - q_s\right)\phi_1^s \pi_s\left(q\right)dq. \tag{27}$$

subject to

$$\sum_s \int_{q_s}^{\bar{q}} \left(q - q_s\right)\left[\phi_1^s \pi_s\left(q\right) - \phi_0^s p_s\left(q\right)\right]dq \geq C. \tag{28}$$

The necessary first-order conditions associated with this program are equation (9) and the binding IC

$$\sum_s \int_{\overline{LR}_s(q)>\kappa} \left(q - q_s^{**}(\kappa)\right)\left[\phi_1^s \pi_s\left(q\right) - \phi_0^s p_s\left(q\right)\right]dq = C, \tag{29}$$

where $\kappa := \frac{\lambda}{\lambda-1}$ and λ is the Lagrange multiplier associated with the IC.

The remainder of the proof follows the same steps as the proof of Lemma 7. Each κ determines $q_s^*(\kappa)$ according to equation (9). From the Intermediate Value Theorem, there exists κ that solves equation (29): the LHS of (29) evaluated at $\kappa = 0$ exceeds C (since $\mathbb{E}\left[q|e=1\right] - \mathbb{E}\left[q|e=0\right] > C$) and it converges to $0 < C$ as $\kappa \to \infty$. Moreover, the firm's profits are ordered by κ: by MLRP, higher thresholds are associated with higher strike prices, which are cheaper. Thus, the best contract among all contracts that satisfy the necessary optimality conditions is the one associated with the largest κ, yielding (10). ■

Proof of Proposition 1. From Lemma 1, there are two possible cases in which the optimal contract does not depend on the signal ($q^*_{s_1} = ... = q^*_{s_S} = q^*$): an interior solution $q^* \in (0, \bar{q})$ and a boundary solution $q^* \in \{0, \bar{q}\}$. Using the conditions from equation (9) for an interior solution establishes:

$$\overline{LR}_{s_i}(q^*) = \overline{LR}_{s_j}(q^*) = \kappa \ \ \forall s_i, s_j, \tag{30}$$

where κ is determined by (10). Using the definition of $\overline{LR}_s(q)$ and rearranging yields the result stated in the proposition.

We now verify that the solution cannot be at the boundary. For a boundary solution we need either $\overline{LR}_s(0) > \kappa$ for all s or $\overline{LR}_s(\bar{q}) < \kappa$ for all s. In the first case, the firm always receives zero, which contradicts the optimality of implementing high effort (since the firm can always obtain strictly positive profits by paying zero in all states and implementing low effort). In the second case, the manager always receives zero, violating equation (10) as the IC is not satisfied. ■

Proof of Lemma 2. For now we ignore the LL (16). Denoting by λ and μ the Lagrange multipliers associated respectively with (14) and (17), the first-order condition ("FOC") with respect to $w_s(q)$ in the program in (13), (14), and (17) is:

$$\begin{aligned} & \phi^s_{\hat{e}} f(q|\hat{e},s) - \lambda \phi^s_{\hat{e}} u'(\bar{W} + w_s(q)) f(q|\hat{e},s) - \mu u'(\bar{W} + w_s(q)) \left[\frac{d\phi^s_{\hat{e}}}{de} f(q|\hat{e},s) + \phi^s_{\hat{e}} f_e(q|\hat{e},s) \right] = 0 \\ \Leftrightarrow \quad & \frac{1}{u'(\bar{W} + w_s(q))} = \lambda + \mu \left[\frac{d\phi^s_{\hat{e}}/de}{\phi^s_{\hat{e}}} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right]. \end{aligned} \tag{31}$$

With limited liability on the manager only, we have $\underline{m}(q) = \bar{W}$ and $\overline{m}(q) = \infty$, using the notations in Jewitt, Kadan, and Swinkels (2008). Using the FOC in (31), the same reasoning as in Proposition 1 in Jewitt, Kadan, and Swinkels (2008) applies for any given signal realization s, so that the optimal contract for a given s is defined implicitly by:

$$\frac{1}{u'(\bar{W} + w_s(q))} = \begin{cases} \lambda + \mu \left[\frac{d\phi^s_{\hat{e}}/de}{\phi^s_{\hat{e}}} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right] & \text{if } \lambda + \mu \left[\frac{d\phi^s_{\hat{e}}/de}{\phi^s_{\hat{e}}} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right] \geq \frac{1}{u'(\bar{W})}, \\ \frac{1}{u'(\bar{W})} & \text{if } \lambda + \mu \left[\frac{d\phi^s_{\hat{e}}/de}{\phi^s_{\hat{e}}} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right] < \frac{1}{u'(\bar{W})}, \end{cases} \tag{32}$$

with $\lambda \geq 0$ and $\mu > 0$. That is,

$$w_s(q) = \begin{cases} u'^{-1}\left(1/\left(\lambda + \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)\right) - \bar{W} & \text{if } \lambda + \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}\right] \geq \frac{1}{u'(\bar{W})} \\ 0 & \text{if } \lambda + \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}\right] < \frac{1}{u'(\bar{W})}, \end{cases} \tag{33}$$

Equation (33) can be rewritten as (18). ■

Proof of Proposition 2.

Because of MLRP, for each s, the optimal contract in equation (18) depends on $LR_s(q)$ for $q \geq q_s^{**}$, where q_s^{**} are defined in equation (20), while it is equal to zero for $q \notin [q_s^{**}, \bar{q}]$. Therefore, if $LR_{s_1}(q) = LR_{s_2}(q)$ for any $q \in [q_{s_1}^{**}, \bar{q}] \cup [q_{s_2}^{**}, \bar{q}]$ and any s_1, s_2, then the payment is independent of s, otherwise it depends on s for some output realizations. ■

Proof of Lemma 3.

We start by characterizing the optimal contract that induces effort $\hat{e}$. For effort e, we have $q \sim \mathcal{N}(h_s(e), \sigma_s^2)$, with $h_s'(e) > 0$ and $h_s(0) = 0$, and we let φ_s be the PDF of the normal distribution with mean zero and standard deviation σ_s.

Letting $W_s(q) = w_s(q) + \bar{W}$ to simplify notation, the manager's IC is:

$$\hat{e} \in \arg\max_{e \in E} \sum_s \phi_e^s \int \ln[W_s(q)]\, \varphi_s(q - h_s(e))\, dq - c(e).$$

The IR and LL are, respectively:

$$\sum_s \phi_{\hat{e}}^s \int \ln[W_s(q)]\, \varphi_s(q - h_s(\hat{e}))\, dq - c(\hat{e}) \geq 0,$$

and

$$W_s(q) \geq \bar{W} \;\forall q, s.$$

To simplify the analysis, we will work with the manager's indirect utility:

$$u_s(q) := \ln[W_s(q)],$$

so that $W_s(q) = \exp[u_s(q)]$. This part is without loss of generality. The next step, which relies on the FOA, is to replace the IC by its FOC:

$$\sum_s \frac{d\phi_{\hat{e}}^s}{de} \int u_s\left(q\right) \varphi_s\left(q - h_s(\hat{e})\right) dq + \sum_s \phi_{\hat{e}}^s \int u_s\left(q\right) h_s'(\hat{e}) \frac{q - h_s(\hat{e})}{\sigma_s^2} \varphi_s\left(q - \hat{e}\right) dq - c'(\hat{e}) \begin{cases} \geq 0 & \text{if } \hat{e} = \bar{e} \\ = 0 & \text{if } \hat{e} \in (0, \bar{e}) \\ \leq 0 & \text{if } \hat{e} = 0 \end{cases}, \tag{34}$$

This FOC rewrites as:

$$\sum_s \int u_s\left(q\right) \varphi_s\left(q - h_s(\hat{e})\right) \left[\frac{d\phi_{\hat{e}}^s}{de} + \phi_{\hat{e}}^s h_s'(\hat{e}) \frac{q - h_s(\hat{e})}{\sigma_s^2}\right] dq - c'(\hat{e}) \begin{cases} \geq 0 & \text{if } \hat{e} = \bar{e} \\ = 0 & \text{if } \hat{e} \in (0, \bar{e}) \\ \leq 0 & \text{if } \hat{e} = 0 \end{cases}. \tag{35}$$

The principal's relaxed program is:

$$\max_{u_s(\cdot)} \sum_s \phi_{\hat{e}}^s \int \left\{q - \exp\left[u_s(q)\right]\right\} \varphi_s\left(q - h_s(\hat{e})\right) dq$$

subject to (34),

$$\sum_s \phi_{\hat{e}}^s \int u_s\left(q\right) \varphi_s\left(q - h_s(\hat{e})\right) dq - c(\hat{e}) \geq 0, \tag{36}$$

and

$$u_s\left(q\right) \geq \ln\left(\bar{W}\right) \ \forall q, s.$$

Lemma 4 *The solution of the relaxed program that implements effort $\hat{e}$ satisfies*

$$W_s(q) = \begin{cases} \bar{W} & \text{for } q \leq h_s(\hat{e}) + \frac{\sigma_s^2}{h_s'(\hat{e})}\left[\frac{\bar{W}-\lambda}{\mu} - \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}\right] \\ \lambda + \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + h_s'(\hat{e})\frac{q-h_s(\hat{e})}{\sigma_s^2}\right] & \text{for } q > h_s(\hat{e}) + \frac{\sigma_s^2}{h_s'(\hat{e})}\left[\frac{\bar{W}-\lambda}{\mu} - \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}\right] \end{cases},$$

where $\lambda \geq 0$, and $\mu > 0$ for $\hat{e} > 0$.

Proof. As usual, the optimal way to implement $\hat{e} = 0$ is to set $W_s(q) = \bar{W}\ \forall s$. To see this, note that $u_s(q) = \ln\left(\bar{W}\right)$ solves the program if we ignore constraint (34). Moreover, $u_s(q) = \ln\left(\bar{W}\right)$ satisfies (34), since

$$\sum_s \frac{d\phi_0^s}{de} \int \ln\left(\bar{W}\right) \varphi_s\left(q\right) dq + \sum_s \phi_0^s \int \ln\left(\bar{W}\right) \frac{q}{\sigma_s^2} \varphi_s\left(q\right) dq - c'(0)$$
$$= \ln\left(\bar{W}\right) \left[\sum_s \frac{d\phi_0^s}{de} \int \varphi_s\left(q\right) dq + \frac{1}{\sigma_s^2} \sum_s \phi_0^s \int q\varphi_s\left(q\right) dq\right] - c'(0) = -c'(0) \leq 0,$$

where we used $\sum_s \frac{d\phi_0^s}{de} = 0$, $\int \varphi_s(q)\,dq = 1$, and $\int q\varphi_s(q)\,dq = 0\ \forall s$.

Next, consider the principal's optimal way to implement a fixed effort $\hat{e} \in (0, \bar{e}]$. The relaxed program maximizes a strictly concave function subject to linear constraints, so the FOC below, the complementary slackness conditions and the constraints are necessary and sufficient. Pointwise optimization gives:

$$\begin{aligned} &-\exp[u_s(q)]\,\phi_{\hat{e}}^s \varphi_s(q - h_s(\hat{e})) + \mu\left[\frac{d\phi_{\hat{e}}^s}{de} + \phi_{\hat{e}}^s h_s'(\hat{e})\frac{q - h_s(\hat{e})}{\sigma_s^2}\right]\varphi_s(q - h_s(\hat{e})) \\ &+\lambda\phi_{\hat{e}}^s \varphi_s(q - h_s(\hat{e})) + \lambda_{LL}(q,s) = 0, \end{aligned}$$

where μ is the multiplier associated with the IC in the relaxed program, while λ is the multiplier associated with IR, and $\lambda_{LL}(q,s)$ are the multipliers associated with the LL. Letting $\tilde{\lambda}_{LL}(q,s) \equiv \frac{\lambda_{LL}(q,s)}{\phi_{\hat{e}}^s\varphi_s(q-h_s(\hat{e}))} \geq 0$, we can rewrite the FOC as:

$$W_s(q) = \lambda + \tilde{\lambda}_{LL}(q,s) + \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + h_s'(\hat{e})\frac{q - h_s(\hat{e})}{\sigma_s^2}\right]. \tag{37}$$

There are two cases to consider. First of all, let $\lambda = 0$. If $\lambda = 0$ at the optimal contract, it can be verified that the following solves the necessary and sufficient optimality conditions:

$$W_s(q) = \begin{cases} \bar{W} & \text{for } q \leq h_s(\hat{e}) + \frac{\sigma_s^2}{h_s'(\hat{e})}\left[\frac{\bar{W}}{\mu} - \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}\right] \\ \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + h_s'(\hat{e})\frac{q-h_s(\hat{e})}{\sigma_s^2}\right] & \text{for } q > h_s(\hat{e}) + \frac{\sigma_s^2}{h_s'(\hat{e})}\left[\frac{\bar{W}}{\mu} - \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}\right] \end{cases}, \tag{38}$$

where μ is chosen so that IC holds (it can be shown that such $\mu > 0$ exists and is unique). To see this, note that when LL binds, we have $W_s(q) = \bar{W}$ and $\tilde{\lambda}_{LL}(q,s) \geq 0$. Then, FOC becomes:

$$\tilde{\lambda}_{LL}(q,s) = \bar{W} - \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + h_s'(\hat{e})\frac{q - h_s(\hat{e})}{\sigma_s^2}\right] - \lambda = \bar{W} - \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + h_s'(\hat{e})\frac{q - h_s(\hat{e})}{\sigma_s^2}\right] \geq 0,$$

which is positive because $q \leq h_s(\hat{e}) + \frac{\sigma_s^2}{h_s'(\hat{e})}\left[\frac{\bar{W}}{\mu} - \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}\right]$. When the LL does not bind ($W_s(q) > \bar{W}$), we have $\tilde{\lambda}_{LL}(q,s) = 0$, so that FOC becomes:

$$W_s(q) = \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + h_s'(\hat{e})\frac{q - h_s(\hat{e})}{\sigma_s^2}\right],$$

where μ is chosen so that IC holds. If the resulting contract satisfies IR in (36), then indeed $\lambda = 0$ at the optimal contract, which is described in (38). This establishes that an option with state-contingent strike price $h_s(\hat{e}) + \frac{\sigma_s^2}{h_s'(\hat{e})}\left[\frac{\bar{W}}{\mu} - \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}\right]$ and sensitivity $n_s^* := \mu\frac{h_s'(\hat{e})}{\sigma_s^2}$ solves the relaxed program when we are implementing $\hat{e} > 0$.

Now suppose that the resulting contract does not satisfy IR in (36). Then we have $\lambda > 0$. It can be verified that the following solves the necessary and sufficient optimality conditions:

$$W_s(q) = \begin{cases} \bar{W} & \text{for } q \le h_s(\hat{e}) + \frac{\sigma_s^2}{h_s'(\hat{e})}\left[\frac{\bar{W}-\lambda}{\mu} - \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}\right] \\ \lambda + \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + h_s'(\hat{e})\frac{q-h_s(\hat{e})}{\sigma_s^2}\right] & \text{for } q > h_s(\hat{e}) + \frac{\sigma_s^2}{h_s'(\hat{e})}\left[\frac{\bar{W}-\lambda}{\mu} - \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}\right] \end{cases},$$

where λ and μ are chosen so that IR and IC hold. To see this, note that when LL binds, we have $W_s(q) = \bar{W}$ and $\tilde{\lambda}_{LL}(q,s) \ge 0$. Then, FOC becomes:

$$\tilde{\lambda}_{LL}(q,s) = \bar{W} - \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + h_s'(\hat{e})\frac{q-h_s(\hat{e})}{\sigma_s^2}\right] - \lambda = \bar{W} - \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + h_s'(\hat{e})\frac{q-h_s(\hat{e})}{\sigma_s^2}\right] - \lambda \ge 0,$$

which is positive because $q \le h_s(\hat{e}) + \frac{\sigma_s^2}{h_s'(\hat{e})}\left[\frac{\bar{W}-\lambda}{\mu} - \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}\right]$. When the LL does not bind ($W_s(q) > \bar{W}$), we have $\tilde{\lambda}_{LL}(q,s) = 0$, so that FOC becomes:

$$W_s(q) = \lambda + \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + h_s'(\hat{e})\frac{q-h_s(\hat{e})}{\sigma_s^2}\right],$$

This establishes that an option with state-contingent strike price $h_s(\hat{e}) + \frac{\sigma_s^2}{h_s'(\hat{e})}\left[\frac{\bar{W}-\lambda}{\mu} - \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}\right]$ and sensitivity $n_s^* := \mu\frac{h_s'(\hat{e})}{\sigma_s^2}$ solves the relaxed program when we are implementing $\hat{e} > 0$. Let $K := \frac{\bar{W}-\lambda}{\mu}$, which is independent from q and s. ■

■

Proof of Proposition 3.

We rely on the proof of Lemma 3.

For point (i), the sensitivity of pay to performance of the option contract is $n_s^* := \mu\frac{h_s'(\hat{e})}{\sigma_s^2}$. In addition, $\frac{d}{dq}\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} = \frac{h_s'(\hat{e})}{\sigma_s^2}$. Therefore, n_s^* is independent of s if and only if $\frac{d}{dq}\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}$ is independent of s, i.e., $\frac{h_s'(\hat{e})}{\sigma_s^2}$ is independent of s.

For point (ii), we can write the optimal contract as:

$$w_s(q) = \max\left\{\lambda + \mu\left(\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + h_s'(\hat{e})\frac{q-h_s(\hat{e})}{\sigma_s^2}\right) - \bar{W}, 0\right\} = \mu\max\left\{LR_s(q) - K, 0\right\}.$$

By construction, the strike price q_s^{**} is such that $w_s(q) > 0$ if and only if $q \ge q_s^{**}$. Therefore, q_s^{**} is independent of s if and only if $LR_s(q) = K$ at the same value of q for all s. ■

B No Monotonicity Constraint

This Appendix considers the core model of Section 1 but without the monotonicity constraint. In Appendix B.1, the only constraint is limited liability on the manager. In Appendix B.2, there is also limited liability on the firm. Appendix B.3 contains proofs for this section.

B.1 Limited Liability on Manager Only

In this subsection, the only constraint is limited liability on the manager. With a continuum of outputs and without limited liability on the firm, existence of an optimal contract is typically an issue.[21] We thus here assume a discrete output distribution $q \in \{q_1, ..., q_Q\}$. Let $\pi_{q,s}$ and $p_{q,s}$ denote the joint probabilities of (q, s) conditional on high and low efforts, respectively (whereas $\pi_s(q)$ and $p_s(q)$ refer to marginal distributions in the core model). To simplify the exposition, we assume full support ($\pi_{q,s} > 0$ and $p_{q,s} > 0$), although this is not needed for our results.

The firm solves the following program:

$$\min_{\{w_{q,s}\}} \sum_{q,s} \pi_{q,s} w_{q,s} \tag{39}$$

$$s.t. \sum_{q,s} \pi_{q,s} w_{q,s} - C \geq 0 \tag{40}$$

$$\sum_{q,s} (\pi_{q,s} - p_{q,s}) w_{q,s} \geq C \tag{41}$$

$$w_{q,s} \geq 0 \quad \forall q, s. \tag{42}$$

As in Section 1, the IC and the manager's LL guarantee that the IR holds. A signal is valuable if including it in the contract (in addition to output) reduces the firm's cost of implementing $e = 1$. Lemma 5 below states that a signal is valuable if and only if it is informative about effort (i.e. affects the likelihood ratio) in states where the payment is strictly positive.

Lemma 5 *Let $\{w_{q,s}\}$ be an optimal contract for implementing $e = 1$ with $w_{q,s_i} > 0$ and $w_{q,s_j} > 0$ for some q, s_i, and s_j. Then, $w_{q,s_i} = w_{q,s_j}$ only if $\frac{\pi_{q,s_i}}{p_{q,s_i}} = \frac{\pi_{q,s_j}}{p_{q,s_j}}$.*

[21] Under discrete outputs, the optimal contract involves the principal paying only in the state with the highest likelihood ratio. With continuous outputs, this is a set of measure zero, so the contract must involve her paying in a neighborhood around that state. Without limited liability, the principal can generically improve on the contract by concentrating the payment in a smaller neighborhood, in which case an optimal contract fails to exist.

Proof of Lemma 5. Fix a vector of payments that satisfy the IC, and consider the following perturbation:

$$w'_{q,s_i} = w_{q,s_i} + \frac{\epsilon}{\pi_{q,s_i} - p_{q,s_i}}, \quad \text{and} \quad w'_{q,s_j} = w_{q,s_j} - \frac{\epsilon}{\pi_{q,s_j} - p_{q,s_j}}.$$

This perturbation keeps the incremental benefit from effort constant and therefore preserves the IC. The LL continues to hold for $\epsilon > 0$ if $w_{q,s_j} > 0$, and for $\epsilon < 0$ if $w_{q,s_i} > 0$. The expected payment (39) increases by:

$$\left(\frac{\pi_{q,s_i}}{\pi_{q,s_i} - p_{q,s_i}} - \frac{\pi_{q,s_j}}{\pi_{q,s_j} - p_{q,s_j}} \right) \epsilon. \tag{43}$$

If the original contract entails $w_{q,s_i} = w_{q,s_j} > 0$ (i.e., a strictly positive payment for output q that does not depend on whether the signal is s_i or s_j), then such a perturbation would satisfy both the IC and LL. Thus, for this contract to be optimal, such a perturbation cannot reduce the expected payment. The term in (43) must be non-positive for all ϵ small enough:

$$\frac{\pi_{q,s_i}}{\pi_{q,s_i} - p_{q,s_i}} = \frac{\pi_{q,s_j}}{\pi_{q,s_j} - p_{q,s_j}},$$

which yields $\frac{\pi_{q,s_i}}{p_{q,s_i}} = \frac{\pi_{q,s_j}}{p_{q,s_j}}$. ■

Lemma 6 states that the payment is strictly positive only in states that maximize the likelihood ratio.

Lemma 6 *Let $\{w_{q,s}\}$ be an optimal contract for implementing $e = 1$. If $\frac{\pi_{q,s_i}}{p_{q,s_i}} < \max_{(q',s')} \left\{ \frac{\pi_{q',s'}}{p_{q',s'}} \right\}$, then $w_{q,s_i} = 0$.*

Combining these results yields Proposition 4, which states that a signal is valuable if and only if it is informative about effort in states with the highest likelihood ratio:

Proposition 4 *A signal has positive value if and only if, $\forall\, (\check{q}, s_j) \in \arg\max_{(q',s')} \{ \frac{\pi_{q',s'}}{p_{q',s'}} \}$, there exists s_k such that $\frac{\pi_{\check{q},s_j}}{p_{\check{q},s_j}} \neq \frac{\pi_{\check{q},s_k}}{p_{\check{q},s_k}}$.*

A signal has positive value if and only if it affects the likelihood ratio at the output level with the maximum likelihood ratio. The firm then increases the payment at the signal where (q, s) has the highest likelihood ratio and decreases it to zero at other signal realizations. In contrast, a signal is not useful if it changes the likelihood ratio only for output levels at which the likelihood ratio is not maximized. Since the payment is zero to begin with, the firm cannot decrease it upon a low signal.

Example 2 below illustrates the result from Proposition 4:

Example 2 *Consider $q \in \{0,1\}$, $s \in \{L,H\}$, and the following conditional probabilities:*

	$e=1$		$e=0$		*Likelihood Ratio*	
	$q=0$	$q=1$	$q=0$	$q=1$	$q=0$	$q=1$
$s=H$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{3}$	2
$s=L$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	2

By Lemma 2, the optimal contract pays only in states $(1,H)$ and $(1,L)$, where the likelihood ratio is maximized. Since the likelihood ratios are equal at these two states, any payments that satisfy the IC with equality generate the same payoff to the firm:

$$\frac{w_{1,H}}{4} + \frac{w_{1,L}}{8} = C.$$

One solution is to pay a payment that does not depend on the signal:

$$w_{1,H} = w_{1,L} = \frac{8}{3}C.$$

Note, however, that q is not a sufficient statistic for e given (q,s) because the likelihood ratios at states $(0,L)$ and $(0,H)$ are different.[22]

With no monotonicity constraint and limited liability on the manager only, it remains the case that some informative signals have zero value. However, the optimal contract is unrealistic – it involves a very large payment in the highest likelihood ratio state, which would typically vastly exceed total output and thus violate a limited liability constraint on the firm, and zero payments in every other state.

[22] It is straightforward to generalize this example to more than two outputs. To see this, let $q \in \{1,...,Q\}$, $\pi_{N,H} = \alpha$, $\pi_{N,L} = \beta$, $p_{N,H} = \frac{\alpha}{2}$, $p_{N,L} = \frac{\beta}{2}$, and $\frac{\pi_{q,s}}{p_{q,s}} < 2$ for all $q \neq N$ and all s. Note that q is not a sufficient statistic for e given (q,s) as long as the likelihood ratio is not constant: $\frac{\pi_{q,H}}{p_{q,H}} \neq \frac{\pi_{q,L}}{p_{q,L}}$ for some q. As before, the optimal contract pays zero in all states except the ones with the highest likelihood ratios: (N,H) and (N,L). Moreover, any wage in these states that satisfies the IC with equality is optimal. In particular, paying $w_{N,H} = w_{N,L} = \frac{2C}{\alpha+\beta}$, $w_{q,H} = w_{q,L} = 0$ for $q \neq N$ is optimal.

B.2 Bilateral Limited Liability

In this subsection, in addition to limited liability on the manager, there is also limited liability on the firm. Thus, the payment cannot exceed q:

$$0 \leq w_s\left(q\right) \leq q \quad \forall q, s. \tag{44}$$

For each s, let q_0^s be determined by $\phi_1^s \pi_s(q_0^s) = \phi_0^s p_s(q_0^s)$ if such q_0^s exists. Otherwise, let $q_0^s = 0$ if $\phi_1^s \pi_s(q) > \phi_0^s p_s(q)$ for all q, and $q_0^s = \bar{q}$ if $\phi_1^s \pi_s(q) < \phi_0^s p_s(q)$ for all q. By MLRP, q_0^s exists and is unique. To ensure that high effort is implementable, we assume:

$$\int_{q_0^s}^{\bar{q}} q \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] dq > C. \tag{45}$$

If (45) were not satisfied, the firm would implement low effort and the optimal contract would trivially involve a zero payment.

Similar to Innes (1990), the solution involves paying the minimum amount possible (zero) when the likelihood ratio is below a threshold κ, and the maximum amount possible when it exceeds it. The threshold κ is chosen so that the IC binds (existence is shown in Appendix B.3); if more than one such threshold exists, we choose the largest one:

$$\kappa := \sup \left\{ \hat{\kappa} : \sum_s \int_{LR_s(q) > \hat{\kappa}} q \left[\phi_1^s \pi_s\left(q\right) - \phi_0^s p_s\left(q\right)\right] dq = C \right\}. \tag{46}$$

By MLRP, for each signal realization, the threshold for the likelihood ratio translates into a threshold for output. Lemma 7 characterizes the optimal contract:

Lemma 7 *The optimal contract under risk neutrality and bilateral limited liability is*

$$w_s(q) = \begin{cases} 0 & \text{if } q < q_s^{***}\left(\kappa\right) \\ q & \text{if } q > q_s^{***}\left(\kappa\right) \end{cases}, \tag{47}$$

where

$$q_s^{***}\left(\kappa\right) := \begin{cases} 0 & \text{if } LR_s(0) > \kappa \\ \bar{q} & \text{if } LR_s(\bar{q}) < \kappa \\ LR_s^{-1}\left(\kappa\right) & \text{if } LR_s(0) \leq \kappa \leq LR_s(\bar{q}) \end{cases} \tag{48}$$

and κ is determined by (46).

Lemma 7 yields a "live-or-die" contract: the manager receives the maximum q if output exceeds a threshold q_s^{***} and zero otherwise. For a given signal realization s, the threshold output level q_s^{***} is chosen so that the likelihood ratio at this output level equals κ.[23] This contract is similar to performance shares, where the the manager receives shares in his firm if and only if performance exceeds a certain threshold, and zero otherwise.

In general, the output threshold will depend on the signal realization s, and so the optimal contract is contingent upon both output and the signal. Proposition 5 gives a condition expressed in terms of model primitives for when the threshold is signal-independent.

Proposition 5 *The optimal contract under risk neutrality and bilateral limited liability is independent of the signal if and only if $LR_s^{-1}(\kappa)$ does not depend on s, where κ is determined by (46).*

If $LR_s^{-1}(\kappa)$ does not depend on s, define $q^{***} := LR_s^{-1}(\kappa)$, and we have:

$$LR_{s_i}(q^{***}) = LR_{s_j}(q^{***}) = \kappa \;\; \forall s_i, s_j. \tag{49}$$

If and only if the output q_s^{***} associated with a likelihood ratio of κ is the same for every s, i.e. $q_s^{***} = q^{***}$, then the firm optimally sets the same threshold q^{***} for all signal realizations, and so the contract is independent of the signal.

The likelihood ratios in Propositions 1 and 5 concern different events. With bilateral limited liability (Proposition 5), the manager is paid q if output exceeds q^{***}. Thus, if the firm uses the signal to vary q^{***}, it changes the payment only in a neighborhood around q^{***} (i.e. changes it from 0 to q or vice-versa). As a result, a signal is only useful if it affects the likelihood ratio at a single point $q = q^{***}$ – i.e. provides information on whether $q = q^{***}$ is more likely to have resulted from working or shirking. If signal realization s_i suggests that the manager has worked, the firm increases the payment from 0 to q by reducing the threshold to $q_{s_i}^{***} < q^{***}$. If it suggests that he has shirked, the firm reduces the payment from q to 0 by increasing the threshold to $q_{s_i}^{***} > q^{***}$.

With a monotonicity constraint (Proposition 1), the manager is paid $(q - q^*)$ if output exceeds q^*. Thus, if the firm uses the signal to vary the strike price q^*, this changes the payment at not only $q = q^*$ (as in Proposition 5) but at all $q \geq q^*$; it cannot change the payment at specific output levels in isolation as this would violate the monotonicity constraint. Thus, a signal has value if it affects the likelihood ratio over a whole range $q \geq q^*$ – i.e. provides

[23] For some signal realizations, this threshold output level may be a corner solution, in which case the manager either always receives the maximum or always receives zero. If all thresholds are interior, then $q_s^{***} = LR_s^{-1}(\kappa)$ for all s.

information on whether $q \geq q^*$ is more likely to have resulted from working or shirking. Any signal that shifts probability mass from below to above the threshold (or vice-versa) is valuable, as it affects the likelihood that output exceeds the threshold. For example, consider $q^* = 5$. The likelihood ratio is higher for $q = 7$ than $q = 3$, and so (in the absence of a signal), the manager receives 2 if $q = 7$ and 0 if $q = 3$. If the event $(q \geq 5, s = s_i)$ indicates effort more than $(q \geq 5, s = s_j)$, i.e., given the knowledge that $q \geq 5$, s_i indicates effort more than s_j, the firm will optimally increase the payment when the signal is s_i compared to when it is s_j. To avoid violating the monotonicity constraint, this is achieved by setting a lower threshold for s_i than for s_j: $q^*_{s_i} < q^*_{s_j}$.

However – as with bilateral limited liability – any signal that only redistributes mass below the threshold so that it stays below the threshold, or only redistributes mass above the threshold so that it stays above the threshold, has no value. Continuing the earlier example, if $(q \geq 7, s = s_i)$ indicates effort more than $(q \geq 7, s = s_j)$, but $(q \geq 5, s = s_i)$ does not indicate effort more than $(q \geq 5, s = s_j)$, then the firm would like to increase the payment for $(q \geq 7, s = s_i)$ and keep unchanged the payment for $(q \geq 5, s = s_i)$. However, such a change would violate the monotonicity constraint, and so the firm would not use the signal.

Despite the difference in the relevant likelihood ratios, Propositions 1 and 5 both establish similar conditions for a signal to have value. In both cases, the firm's only degree of freedom is the threshold q^* or q^{***} – under the optimal contract, the payment below the threshold is zero, and the payment above is either the entire output or the residual. Thus, an additional signal will only be included if the firm wishes to use its realization to vary the threshold – it will not use it to change any other dimension of the contract. With bilateral limited liability, changing q^{***} only has local effects, and so Proposition 5 depends on the likelihood ratio associated with $q = q^{***}$. With a monotonicity constraint, changing q^* affects payments at all higher outputs, and so Proposition 1 depends on the likelihood ratio associated with $q \geq q^*$. Overall, with limited liability on the firm rather than a monotonicity constraint, it remains the case that some informative signals have zero value.

B.3 Proofs

Proof of Lemma 6. Let $(\check{q}, s_j) \in \arg\max_{(q'', s'')} \left\{ \frac{\pi_{q'', s''}}{p_{q'', s''}} \right\}$ denote a state with the highest likelihood ratio and consider a state (q, s_i) that does not have the highest likelihood ratio:

$$\frac{\pi_{q,s_i}}{p_{q,s_i}} < \frac{\pi_{\check{q},s_j}}{p_{\check{q},s_j}}. \tag{50}$$

Consider the following perturbation, which, as in the proof of Lemma 5, keeps the incremental benefit from effort constant, thereby preserving the IC:

$$w'_{q,s_i} = w_{q,s_i} - \frac{\epsilon}{\pi_{q,s_i} - p_{q,s_i}}, \quad \text{and} \quad w'_{\check{q},s_j} = w_{\check{q},s_j} + \frac{\epsilon}{\pi_{\check{q},s_j} - p_{\check{q},s_j}}.$$

LL continues to hold for $\epsilon > 0$ if $w_{q,s_i} > 0$ and for $\epsilon < 0$ if $w_{\check{q},s_j} > 0$. The expected payment (39) increases by:

$$\left(\frac{\pi_{\check{q},s_j}}{\pi_{\check{q},s_j} - p_{\check{q},s_j}} - \frac{\pi_{q,s_i}}{\pi_{q,s_i} - p_{q,s_i}}\right)\epsilon. \tag{51}$$

From (50), the term inside the parentheses in (51) is strictly negative. Thus, the firm can reduce the expected payment by selecting $\epsilon > 0$ small enough, which does not violate the LL when $w_{q,s_i} > 0$. As a result, the solution entails zero payments in all states that do not maximize the likelihood ratio. ■

Proof of Lemma 7. The firm's program is:

$$\min_{\{w_s(q)\}} \sum_s \int_0^{\bar{q}} w_s(q)\, \phi_1^s \pi_s(q)\, dq$$

subject to

$$0 \le w_s(q) \le q \ \ \forall q \in [0, \bar{q}],$$
$$\sum_s \int_0^{\bar{q}} w_s(q) \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] dq \ge C.$$

This is an infinite-dimensional linear program, which has the following first-order conditions:

$$w_s(q) = \left\{ \begin{array}{c} q \\ 0 \end{array} \right\} \text{ if } \phi_1^s \pi_s(q) - \mu \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] \left\{ \begin{array}{c} > \\ < \end{array} \right\} 0, \tag{52}$$

for all s (where μ is the Lagrange multiplier associated with the IC), as well as the IC, which must bind:

$$\sum_s \int_{LR_s(q) \ge \frac{\mu}{\mu-1}} q \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] dq = C. \tag{53}$$

Letting $\kappa := \frac{\mu}{\mu-1}$ and using (52), it follows that $w_s(q) = q$ if $LR_s(q) > \kappa$, and $w_s(q) = 0$ if

$LR_s(q) < \kappa$. Moreover, equation (53) becomes:

$$\sum_s \int_{LR_s(q)>\kappa} q[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)]dq = C. \tag{54}$$

We first show that the set of contracts satisfying these necessary conditions is not empty. Since each value of κ fully characterizes a contract through equations (47) and (48), it suffices to show that there exists a κ that solves (54). The left-hand side ("LHS") of (54) converges to $\int_{q_0^s}^{\bar{q}} q\left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] dq$ as $\kappa \searrow 1$. From (45), this exceeds C. Moreover, it converges to $0 < C$ as $\kappa \nearrow +\infty$. Therefore, by the Intermediate Value Theorem, there exists κ satisfying (54).

Notice that κ orders all contracts that satisfy the necessary optimality conditions: by MLRP, a higher threshold for the likelihood ratio means that the firm pays (weakly) less in each state. Thus, if (54) has multiple solutions, the optimum is the contract associated with the highest κ, as defined in equation (46). ■

Proof of Proposition 5. From Lemma 7, there are two possible cases in which the optimal contract does not depend on the signal ($q_{s_1}^* = ... = q_{s_S}^* = q^*$): an interior solution $q^* \in (0, \bar{q})$ and a boundary solution $q^* \in \{0, \bar{q}\}$. Using the conditions from Lemma 7 for an interior solution establishes:

$$LR_{s_i}(q^{***}) = LR_{s_j}(q^{***}) = \kappa \ \ \forall s_i, s_j, \tag{55}$$

where κ is determined by (46). Using the definition of $LR_s(q)$ and rearranging yields the result stated in the proposition.

We now verify that the solution cannot be at the boundary. For a boundary solution we need either $LR_s(0) > \kappa$ for all s or $LR_s(\bar{q}) < \kappa$ for all s. In the first case, the firm always receives zero, which contradicts the optimality of implementing high effort (since the firm can always obtain strictly positive profits by paying zero in all states and implementing low effort). In the second case, the manager always receives zero, violating equation (46) as the IC is not satisfied. ■

Supplementary Appendix for
"The Value of Performance Signals Under Limited Liability"

The first-order approach with limited liability, normally-distributed output, and log utility

This Appendix provides sufficient conditions for the FOA in the setting considered in Section 2.1, with limited liability, normally-distributed output, and log utility. We first derive the optimal contract and provide a sufficient condition for the FOA without an additional signal. Given effort $e \in [0, \bar{e}]$, output is determined by

$$q = e + \epsilon,$$

where $\epsilon \sim \mathcal{N}(0, \sigma^2)$.

Proposition 6 *Suppose $C''(e) \geq \frac{\bar{e}}{\sigma^2}$ for all $e \in [0, \bar{e}]$. Let $\{w^*(\cdot), e^*\}$ be the optimal compensation contract and the effort it implements. Then, there exists $\lambda > 0$ and $q^* \leq e^* + \frac{\sigma^2}{\lambda}\bar{W}$ such that*

$$w^*(q) = \frac{\lambda}{\sigma^2} \cdot \max\{q - q^*,\ 0\}.$$

Moreover, $q^* = e^* + \frac{\sigma^2}{\lambda}\bar{W}$ if the IR does not bind.

For example, with a quadratic effort cost, $C(e) = \alpha e + \frac{\beta}{2}e^2$, for $\alpha > 0$ and $\beta > 0$, we have $C''(e) = \beta$ for all e, and the condition for the validity of the FOA is simply $\beta \geq \frac{\bar{e}}{\sigma^2}$.

Proof of Proposition 6:

As usual, let φ denote the PDF of the standard normal distribution. Let $W(q) \equiv w(q) + \bar{W}$ denote the manager's consumption (i.e., the manager's initial wealth $\bar{W}$ plus his pay). The manager's IC is:

$$e \in \arg\max_{\hat{e} \in [0, \bar{e}]} \int \ln[W(q)] \frac{1}{\sigma}\varphi\left(\frac{q - \hat{e}}{\sigma}\right) dq - C(\hat{e}).$$

The IR and LL are, respectively:

$$\int \ln[W(q)] \frac{1}{\sigma}\varphi\left(\frac{q - e}{\sigma}\right) dq - C(e) \geq 0,$$

and

$$W(q) \geq \bar{W} \ \forall q.$$

To simplify the notation, we will work with the manager's indirect utility:

$$u(q) \equiv \ln\left[W(q)\right],$$

so that $W(q) = \exp\left[u(q)\right]$. This part is without loss of generality. The next step, which in general is not without loss of generality, is that we will replace the IC by its FOC:

$$\int u(q)\left(\frac{q-e}{\sigma^3}\right)\varphi\left(\frac{q-e}{\sigma}\right)dq - C'(e)\begin{cases} \geq 0 & \text{if } e=\bar{e} \\ =0 & \text{if } e\in(0,\bar{e}) \\ \leq 0 & \text{if } e=0 \end{cases}, \tag{56}$$

where we used the fact that $\varphi'(q) = -x\varphi(q)$, so that $\frac{d}{de}\left[\varphi\left(\frac{q-e}{\sigma}\right)\right] = \frac{q-e}{\sigma^2}\cdot\varphi\left(\frac{q-e}{\sigma}\right)$. Since replacing the IC by its FOC is not always justified, after solving the firm's relaxed program (in which we replace IC by its FOC), we will need to verify that its solution satisfies the IC.

Writing in terms of the manager's indirect utility, the IR becomes

$$\int u(q)\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)dq - C(e) \geq 0. \tag{57}$$

It is also convenient to multiply both sides of LL by $\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right) > 0$, rewriting it as:

$$\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)u(q) \geq \frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)\ln\left(\bar{W}\right) \ \forall q. \tag{58}$$

The firm's *relaxed program* is:

$$\max_{u(\cdot),e}\int\left\{q-\exp\left[u(q)\right]\right\}\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)dq$$

subject to (56), (58), and (57).

As in Grossman and Hart (1983), we break down this program in two parts. First, we consider the solution of the relaxed program holding each effort level $e\in[0,\bar{e}]$ fixed:

$$\min_{u(\cdot)}\int\exp\left[u(q)\right]\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)dq$$

subject to (56), (58), and (57).

The optimal contract to implement the lowest effort ($e^* = 0$) in the relaxed program (as well as in the original program) pays a fixed wage. The utility given to the manager is set at the lowest level that still satisfies both LL and IR: $u(q) = \max\{\ln(\bar{W}), C(0)\}$ for all q. To see this, notice that a constant utility $u(q) = u^*$ always satisfies (56):

$$\int u^* \frac{q}{\sigma^3} \varphi\left(\frac{q}{\sigma}\right) dq - C'(0) = \frac{u^*}{\sigma^3} \times \int q\varphi\left(\frac{q}{\sigma}\right) dq - C'(0) = -C'(0) \leq 0.$$

The next lemma obtains the solution of the relaxed program for $e^* > 0$.

Lemma 8 *The optimal contract that implements $e^* > 0$ in the relaxed program is:*

$$w(q) = \frac{\lambda}{\sigma^2} \cdot \max\{q - q^*, 0\},$$

where $q^* \leq e^* + \frac{\sigma^2}{\lambda}\bar{W}$ (with equality if the IR does not bind).

Proof. The (infinite-dimensional) Lagrangian gives the following FOC:

$$-\exp[u(q)] \frac{1}{\sigma}\varphi\left(\frac{q-e^*}{\sigma}\right) + \lambda\left(\frac{q-e^*}{\sigma^3}\right)\varphi\left(\frac{q-e^*}{\sigma}\right) + \mu_{IR}\frac{1}{\sigma}\varphi\left(\frac{q-e^*}{\sigma}\right) + \mu_{LL}(q)\frac{1}{\sigma}\varphi\left(\frac{q-e^*}{\sigma}\right) = 0,$$

where λ is the multiplier associated with (56), and μ_{LL} and μ_{IR} are the multipliers associated with (58), and (57). Since the program corresponds to the minimization of a strictly convex function subject to linear constraints, the FOC above, along with the standard complementary slackness conditions and the constraints, are sufficient for an optimum. Substitute $\exp[u(q)] = W(q)$ and simplify the FOC above to obtain:

$$W(q) = \lambda\left(\frac{q-e^*}{\sigma^2}\right) + \mu_{IR} + \mu_{LL}(q).$$

Suppose first that IR doesn't bind so that $\mu_{IR} = 0$. Then, the FOC becomes

$$W(q) = \lambda\left(\frac{q-e^*}{\sigma^2}\right) + \mu_{LL}(q).$$

For $W(q) > \bar{W}$, complementary slackness gives $\mu_{LL}(q) = 0$, so that:

$$W(q) = \lambda \times \frac{q-e^*}{\sigma^2},$$

Which exceeds $\bar{W}$ if and only if

$$\lambda \times \frac{q-e^*}{\sigma^2} > \bar{W} \iff q > e^* + \frac{\sigma^2 \bar{W}}{\lambda} \equiv q^*.$$

For $W(q) = \bar{W}$, the FOC becomes:

$$\bar{W} = \lambda \left(\frac{q-e^*}{\sigma^2} \right) + \mu_{LL}(q) \therefore \mu_{LL}(q) = \bar{W} - \lambda \left(\frac{q-e^*}{\sigma^2} \right),$$

so that $\mu_{LL}(q) \geq 0$ if and only if

$$\bar{W} \geq \lambda \times \frac{q-e^*}{\sigma^2} \iff q \leq q^*.$$

Therefore, the optimal contract is

$$W(q) = \max \left\{ \frac{\lambda \left(q-e^* \right)}{\sigma^2}, \ \bar{W} \right\} = \begin{cases} \frac{\lambda(q-e^*)}{\sigma^2} & \text{if } q \geq q^* \\ \bar{W} & \text{if } q \leq q^* \end{cases}.$$

Writing in terms of the firm's payments, we have

$$w(q) = W(q) - \bar{W} = \frac{\lambda}{\sigma^2} \max \left\{ q - q^*, \ 0 \right\},$$

where the last equality uses the definition of q^*. That is, the firm gives the manager an option with strike price $q^* = e^* + \frac{\sigma^2}{\lambda} \bar{W} > e^*$ and a sensitivity $\frac{\lambda}{\sigma^2}$ chosen so that (56) holds (which can be shown to exist and be unique).

Next, suppose that IR binds so that $\mu_{IR} \geq 0$. Then, for $W(q) > \bar{W}$, we must have

$$W(q) = \lambda \left(\frac{q-e^*}{\sigma^2} \right) + \mu_{IR},$$

so that

$$W(q) > \bar{W} \iff \mu_{IR} > \bar{W} - \lambda \left(\frac{q-e^*}{\sigma^2} \right).$$

For $W(q) = \bar{W}$, we obtain:

$$\bar{W} = \lambda \left(\frac{q-e^*}{\sigma^2} \right) + \mu_{IR} + \mu_{LL}(q),$$

so that $\mu_{LL}(q) \geq 0$ if and only if

$$\mu_{LL}(q) = \bar{W} - \lambda\left(\frac{q-e^*}{\sigma^2}\right) - \mu_{IR} \geq 0$$

$$\iff \bar{W} - \lambda\left(\frac{q-e^*}{\sigma^2}\right) \geq \mu_{IR}.$$

Define the strike price q^* as the solution to

$$\bar{W} - \lambda\left(\frac{q^*-e^*}{\sigma^2}\right) = \mu_{IR},$$

that is,

$$q^* \equiv e^* + \frac{\sigma^2}{\lambda}\left(\bar{W} - \mu_{IR}\right) \leq e^* + \frac{\sigma^2}{\lambda}\bar{W}.$$

Then, combining the conditions, we obtain

$$W(q) = \begin{cases} \frac{\lambda}{\sigma^2}(q-q^*) + \bar{W} & \text{if } q \geq q^* \\ \bar{W} & \text{if } q \leq q^* \end{cases},$$

which again corresponds to an option with strike price q^* and sensitivity $\frac{\lambda}{\sigma^2}$. Here, λ and q^* are chosen so that both (56) and (57) hold with equality. ■

We now obtain an upper bound on λ:

Lemma 9 *Suppose $e^* > 0$ is the effort that solves the firm's relaxed program. Then the optimal contract is*

$$w(q) = \max\left\{\frac{\lambda}{\sigma^2}(q-q^*),\ 0\right\},$$

where $0 < \lambda < \sqrt{2\pi}\sigma e^*$ and $q^* \leq e^* + \frac{\sigma^2}{\lambda}\bar{W}$.

Proof. From the previous lemma, we need to show that $\lambda \leq \sqrt{2\pi}\sigma e^*$. Recall that the optimal way to implement effort $e > 0$ is to pay the option:

$$w(q) = \max\left\{\frac{\lambda}{\sigma^2}(q-q^*),\ 0\right\},$$

where $q^* \leq \frac{\sigma^2}{\lambda}\bar{W} + e$. Since the firm's profits are increasing in the strike price q^* (holding all other variables, including effort, constant), her profits are bounded above by the profits from offering the option with the highest strike price ($\bar{q} = \frac{\sigma^2}{\lambda}\bar{W} + e \geq q^*$), which equal

$$e - \left[\frac{\lambda}{\sigma^2}\int_{\frac{\sigma^2}{\lambda}\bar{W}+e}^{\infty}\left(q - e - \frac{\sigma^2}{\lambda}\bar{W}\right)\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)dq\right].$$

Let $z \equiv q - e - \frac{\sigma^2}{\lambda}\bar{W}$, so that $q = z + e + \frac{\sigma^2}{\lambda}\bar{W}$. Note that $q \geq \frac{\sigma^2}{\lambda}\bar{W} + e$ if and only if $z \geq 0$. Thus, we can rewrite this expression as

$$e - \left[\frac{\lambda}{\sigma^2}\int_0^{\infty} z\frac{1}{\sigma}\varphi\left(\frac{z + \frac{\sigma^2}{\lambda}\bar{W}}{\sigma}\right)dz\right].$$

Moreover, since $\varphi(z)$ is decreasing in z for $z > 0$, it follows that

$$\varphi\left(\frac{z + \frac{\bar{W}}{\lambda}}{\sigma}\right) < \varphi\left(\frac{z}{\sigma}\right) \quad \forall z > 0.$$

Thus, the firm's profits are strictly less than

$$e - \frac{\lambda}{\sigma^2}\int_0^{\infty}\frac{z}{\sigma}\varphi\left(\frac{z}{\sigma}\right)dz.$$

Apply the following change of variables $y = \frac{z}{\sigma}$ (so that $z = \sigma y$, $dz = \sigma dy$) to write

$$\int_0^{\infty}\frac{z}{\sigma}\varphi\left(\frac{z}{\sigma}\right)dz = \sigma\int_0^{\infty} y\varphi\left(y\right)dy.$$

Integration by parts, gives

$$\int_0^{\infty} y\varphi(y)dy = [-\varphi(y)]_0^{\infty} = \varphi(0) = \frac{1}{\sqrt{2\pi}}.$$

Substituting in the formula from before, it follows that the firm's profits are strictly less than

$$e - \frac{1}{\sqrt{2\pi}}\cdot\frac{\lambda}{\sigma}.$$

Since the firm can always obtain a profit of zero by paying zero wages and implementing zero effort, we must have

$$e - \frac{1}{\sqrt{2\pi}}\cdot\frac{\lambda}{\sigma} > 0 \iff \lambda < \sqrt{2\pi}\sigma e.$$

■

The following lemma provides an additional upper bound:

Lemma 10 *For any* $q^* \in \mathbb{R}$, $e \in [0, \bar{e}]$, $\sigma > 0$ *and* $\lambda > 0$, *we have*

$$\int \ln \left[\bar{W} + \frac{\lambda}{\sigma^2} \cdot \max \{ (q - q^*) ,\ 0 \} \right] \left[\left(\frac{q-e}{\sigma} \right)^2 - 1 \right] \frac{1}{\sigma} \varphi \left(\frac{q-e}{\sigma} \right) dq$$

$$\leq \int \left[\bar{W} + \frac{\lambda}{\sigma^2} \cdot \max \{ (q - q^*) ,\ 0 \} \right] \left[\left(\frac{q-e}{\sigma} \right)^2 - 1 \right] \frac{1}{\sigma} \varphi \left(\frac{q-e}{\sigma} \right) dq.$$

Proof. For notational simplicity, let $y \equiv \frac{q^*-e}{\sigma}$, apply the change of variables $z \equiv \frac{q-e}{\sigma}$, and let

$$g(z) \equiv \bar{W} + \frac{\lambda}{\sigma} \cdot \max \{ z - y,\ 0 \} - \ln \left[\bar{W} + \frac{\lambda}{\sigma} \cdot \max \{ z - y,\ 0 \} \right].$$

Then, the inequality in the lemma can be written as

$$\int_{-\infty}^{\infty} g(z) \left(z^2 - 1 \right) \varphi (z) \, dz \geq 0.$$

We claim that $g(\cdot)$ is non-decreasing. To see this, notice that, for $z \leq y$, $g(z) = \bar{W} - \ln \bar{W}$ (which is constant in z). For $z > y$, we have

$$g'(z) = \frac{\lambda}{\sigma} \left(\frac{\bar{W} - 1 + \frac{\lambda}{\sigma} (z - y)}{\bar{W} + \frac{\lambda}{\sigma} (z - y)} \right),$$

which is positive for all $z > y$ since $\bar{W} \geq 1$. Because g is non-decreasing, we have $g(q) \geq g(-q)$ for $q \geq 0$ and $\frac{d}{dq} [g(q) - g(-q)] \geq 0$. Note that, applying the change of variables $\tilde{z} = -z$ and using the symmetry of $(z^2 - 1) \varphi (z)$ around zero, we have:

$$\int_{-\infty}^{0} g(z) \left(z^2 - 1 \right) \varphi (z) \, dz = - \int_{0}^{\infty} g(-z) \left(z^2 - 1 \right) \varphi (z) \, dz. \tag{59}$$

Therefore,

$$\begin{aligned}
\int g(z)\left(z^2-1\right)\varphi\left(z\right)dz &= \int_{-\infty}^{0} g(z)\left(z^2-1\right)\varphi\left(z\right)dz + \int_0^\infty g(z)\left(z^2-1\right)\varphi\left(z\right)dz \\
&= -\int_0^\infty g(-z)\left(z^2-1\right)\varphi\left(z\right)dz + \int_0^\infty g(z)\left(z^2-1\right)\varphi\left(z\right)dz \\
&= \int_0^\infty \left[g(z)-g(-z)\right]\left(z^2-1\right)\varphi\left(z\right)dz \\
&= \int_0^1 \left[g(z)-g(-z)\right]\left(z^2-1\right)\varphi\left(z\right)dz + \int_1^\infty \left[g(z)-g(-z)\right]\left(z^2-1\right)\varphi\left(z\right)dz \\
&\geq \int_0^1 \left[g(1)-g(-1)\right]\left(z^2-1\right)\varphi\left(z\right)dz + \int_1^\infty \left[g(1)-g(-1)\right]\left(z^2-1\right)\varphi\left(z\right)dz \\
&= \left[g(1)-g(-1)\right]\int_0^\infty \left(z^2-1\right)\varphi\left(z\right)dz = 0,
\end{aligned}$$

where the first line opens the integral between positive and negative values of z, the second line substitutes (59), the third line combines the terms from the two integrals, and the fourth line opens the integral between $z \leq 1$ and $z \geq 1$. The fifth line is the crucial step, which uses the following two facts: (i) $z^2 > (<)1$ for $z > (<)1$, and (ii) $g(z) - g(-z)$ is non-decreasing for all z. Therefore, substituting $g(z) - g(-z)$ by its upper bound where the term inside the integral is negative and by its lower bound where it is positive lowers the value of the integrand. The sixth line then combines terms and uses the fact that

$$\int_0^\infty \left(z^2-1\right)\varphi\left(z\right)dz = \left[-z\varphi(z)\right]_0^\infty = 0.$$

■

The final lemma shows that the solution of the relaxed program also solves the firm's program if the effort cost is sufficiently convex (the FOA is valid).

Lemma 11 *Suppose $C''(e) \geq \frac{\bar{e}}{\sigma}$ for all $e \in [0, \bar{e}]$.* Then, *the solution of the firm's program coincides with the solution of the relaxed program.*

Proof. The manager's utility from picking effort e is:

$$U(e;\ q^*, \lambda) \equiv \int \ln\left[\bar{W} + \frac{\lambda}{\sigma^2}\cdot \max\left\{\left(q-q^*\right),\ 0\right\}\right]\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)dq - C(e).$$

We know from previous results that $0 < \lambda < \sqrt{2\pi}\sigma e^*$. The FOA is justified if

$$\frac{\partial^2 U}{\partial e^2}(e;\ q^*, \lambda) \leq 0$$

for all $e \in [0, \bar{e}]$, all $q^* \in \mathbb{R}$, and $\lambda \in (0, \sqrt{2\pi}\sigma\bar{e})$.

Differentiation gives

$$\begin{aligned}\frac{\partial^2 U}{\partial e^2} &= \int \ln\left[\bar{W} + \tfrac{\lambda}{\sigma^2}\cdot \max\left\{(q-q^*)\,,\ 0\right\}\right] \tfrac{1}{\sigma}\tfrac{d^2}{de^2}\left[\varphi\left(\tfrac{q-e}{\sigma}\right)\right] dq - C''(e) \\ &= \tfrac{1}{\sigma^2}\int \ln\left[\bar{W} + \tfrac{\lambda}{\sigma^2}\cdot \max\left\{(q-q^*)\,,\ 0\right\}\right] \tfrac{1}{\sigma}\left[\left(\tfrac{q-e}{\sigma}\right)^2 - 1\right]\varphi\left(\tfrac{q-e}{\sigma}\right) dq - C''(e)\end{aligned}\ , \tag{60}$$

where the second line uses the fact that $\frac{d^2}{de^2}\left[\varphi\left(\frac{q-e}{\sigma}\right)\right] = \frac{1}{\sigma^2}\left[\left(\frac{q-e}{\sigma}\right)^2 - 1\right]\varphi\left(\frac{q-e}{\sigma}\right)$. But notice that

$$\begin{aligned} & \int \ln\left[\bar{W} + \tfrac{\lambda}{\sigma^2}\cdot \max\left\{(q-q^*)\,,\ 0\right\}\right]\left[\left(\tfrac{q-e}{\sigma}\right)^2 - 1\right]\tfrac{1}{\sigma}\varphi\left(\tfrac{q-e}{\sigma}\right) dq \\ \leq\ & \int \left[\bar{W} + \tfrac{\lambda}{\sigma^2}\cdot \max\left\{(q-q^*)\,,\ 0\right\}\right]\left[\left(\tfrac{q-e}{\sigma}\right)^2 - 1\right]\tfrac{1}{\sigma}\varphi\left(\tfrac{q-e}{\sigma}\right) dq \\ =\ & \tfrac{\lambda}{\sigma^2}\cdot \int \left[\max\left\{(q-q^*)\,,\ 0\right\}\right]\left[\left(\tfrac{q-e}{\sigma}\right)^2 - 1\right]\tfrac{1}{\sigma}\varphi\left(\tfrac{q-e}{\sigma}\right) dq \\ =\ & \tfrac{\lambda}{\sigma^2}\cdot \int_{q^*}^{\infty} (q-q^*)\left[\left(\tfrac{q-e}{\sigma}\right)^2 - 1\right]\tfrac{1}{\sigma}\varphi\left(\tfrac{q-e}{\sigma}\right) dq, \end{aligned}$$

where the inequality on the second line uses the result from the previous lemma, the third line follows from the fact that $\int\left[\left(\frac{q-e}{\sigma}\right)^2 - 1\right]\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right) dq = 0$ (a Standard Normal variable has variance 1), and the fourth line opens the max operator. Substituting in the expression from (60), we obtain the following sufficient condition for the validity of FOA:

$$\frac{\lambda}{\sigma^4}\cdot \int_{q^*}^{\infty} (q-q^*)\left[\left(\frac{q-e}{\sigma}\right)^2 - 1\right]\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right) dq \leq C''(e) \tag{61}$$

for all $e \in [0, \bar{e}]$, $q^* \in \mathbb{R}$, and $\lambda \in (0, \sqrt{2\pi}\sigma\bar{e})$.

Let $\xi(q^*) \equiv \int_{q^*}^{\infty} (q-q^*)\left[\left(\frac{q-e}{\sigma}\right)^2 - 1\right]\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right) dq$. We claim that $\xi'^*) \left\{\begin{array}{c} > \\ < \end{array}\right\} 0 \iff q^* \left\{\begin{array}{c} < \\ > \end{array}\right\} e$. Differentiation, gives:

$$\xi'^*) = -\int_{q^*}^{\infty}\left[\left(\frac{q-e}{\sigma}\right)^2 - 1\right]\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right) dq. \tag{62}$$

However, note that

$$\frac{d}{dq}\left[-\left(\frac{q-e}{\sigma}\right)\varphi\left(\frac{q-e}{\sigma}\right)\right] = -\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right) - \left(\frac{q-e}{\sigma}\right)\frac{1}{\sigma}\varphi'\left(\frac{q-e}{\sigma}\right) = \left[\left(\frac{q-e}{\sigma}\right)^2 - 1\right]\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right),$$

where the last equality uses the fact that $\varphi'(q) = -q\varphi(q)$. Therefore,

$$\int\left[\left(\frac{q-e}{\sigma}\right)^2 - 1\right]\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right) dq = -\left(\frac{q-e}{\sigma}\right)\varphi\left(\frac{q-e}{\sigma}\right).$$

Substituting back in (62), gives

$$\xi'^{*}) = -\left(\frac{q^{*}-e}{\sigma}\right)\varphi\left(\frac{q^{*}-e}{\sigma}\right)\left\{\begin{array}{c}>\\<\end{array}\right\}0 \iff q^{*}\left\{\begin{array}{c}<\\>\end{array}\right\}e.$$

Therefore, $\xi(\cdot)$ is maximized at $q^{*}=e$, so that, by condition (61), it suffices to show that

$$\frac{\lambda}{\sigma^{4}}\cdot\xi(e)\leq C''(e). \tag{63}$$

Evaluating ξ at e, gives:

$$\xi(e)=\int_{e}^{\infty}(q-e)\left[\left(\frac{q-e}{\sigma}\right)^{2}-1\right]\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)dq.$$

Performing the change of variables $z\equiv\frac{q-e}{\sigma}$, we obtain

$$\xi(e)=\int_{e}^{\infty}\left(\frac{q-e}{\sigma}\right)\left[\left(\frac{q-e}{\sigma}\right)^{2}-1\right]\varphi\left(\frac{q-e}{\sigma}\right)dq=\sigma\int_{0}^{\infty}z\left(z^{2}-1\right)\varphi\left(z\right)dz. \tag{64}$$

Integration by parts, gives

$$\int z\left(z^{2}-1\right)\varphi\left(z\right)dz=-z^{2}\varphi(z)+\int z\varphi\left(z\right)dz,$$

where we let $(z^{2}-1)\varphi\left(z\right)dz=dv$ so that $v=-z\varphi\left(z\right)$, and we let $u=z$, so that $du=dz$. Therefore

$$\int_{0}^{\infty}z\left(z^{2}-1\right)\varphi\left(z\right)dz=\int_{0}^{\infty}z\varphi\left(z\right)dz.$$

Using the fact that $\frac{d}{dz}\left[-\varphi(z)\right]=z\varphi\left(z\right)$, it follows that

$$\int_{0}^{\infty}z\left(z^{2}-1\right)\varphi\left(z\right)dz=\left[-\varphi(z)\right]_{0}^{+\infty}=\varphi(0)=\frac{1}{\sqrt{2\pi}}.$$

Substituting in (64), yields

$$\xi(e)=\frac{\sigma}{\sqrt{2\pi}}.$$

Substituting in condition (63), we obtain the following sufficient condition:

$$\frac{\lambda}{\sqrt{2\pi}\sigma^3} \leq C''(e),$$

which is true for all $e \in [0, \bar{e}]$ and all $\lambda \in (0, \sqrt{2\pi}\sigma\bar{e})$ if and only if

$$C''(e) \geq \frac{\bar{e}}{\sigma^2} \quad \forall e \in [0, \bar{e}].$$

■

The next result provides a sufficient condition for the FOA with an additional signal of performance, for a subset of signal distributions.

Proposition 7 *We consider the same setting as in Proposition 3, and a signal distribution such that: (i) $h_s''(e) \leq 0$ for all s; (ii) ϕ_e^s linear in e for all s; (iii) $h_{s_1}(e) \leq h_{s_2}(e)$, $h'_{s_1}(e) \leq h'_{s_2}(e)$, and $\sigma_{s_1} \geq \sigma_{s_2}$ for any s_1, s_2 with $\frac{d\phi_e^{s_1}}{de} > 0 > \frac{d\phi_e^{s_2}}{de}$ and any $e \in [0, \bar{e}]$. Then the FOA is valid if $C''(e) \geq \sum_s \phi_{e^*}^s h_s(\bar{e}) \frac{\sum_s \frac{\phi_e^s}{\sigma_s^3}(h'_s(e))^2}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s}}$ for all $e \in [0, \bar{e}]$.*

Proof of Proposition 7: Let φ denote the PDF of the Standard Normal distribution. Let $W_s(q) := \bar{W} + w_s(q)$ denote the manager's consumption (i.e., the manager's initial wealth $\bar{W}$ plus his pay).

The manager's IC, IR, and LL, are, respectively:

$$e \in \arg\max_{\hat{e} \in [0,\bar{e}]} \sum_s \phi_e^s \int \ln [W_s(q)] \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(\hat{e})}{\sigma_s}\right) dq - C(\hat{e}),$$

$$\sum_s \phi_e^s \int \ln [W_s(q)] \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e)}{\sigma_s}\right) dq - C(e) \geq 0,$$

and

$$W_s(q) \geq \bar{W} \ \forall q, s.$$

To simplify notation, we will work with the manager's indirect utility:

$$u_s(q) := \ln [W_s(q)],$$

so that $W_s(q) = \exp[u_s(q)]$. We replace the IC by its FOC (we verify the validity of the FOA below):

$$\sum_s \phi_e^s \int u_s(q) \frac{q - h_s(e)}{\sigma_s^3} \varphi\left(\frac{q - h_s(e)}{\sigma_s}\right) dq - C'(e) \begin{cases} \geq 0 & \text{if } \ e = \bar{e} \\ = 0 & \text{if } \ e \in (0, \bar{e}) \\ \leq 0 & \text{if } \ e = 0 \end{cases} . \tag{65}$$

We also multiply both sides of LL by $\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)\phi_e^s > 0$, rewriting it as:

$$\frac{1}{\sigma_s}\varphi\left(\frac{q - h_s(e)}{\sigma_s}\right)\phi_e^s u_s(q) \geq \frac{1}{\sigma_s}\varphi\left(\frac{q - h_s(e)}{\sigma_s}\right)\phi_e^s \ln\left(\bar{W}\right) \ \ \forall q, s. \tag{66}$$

The firm's *relaxed program* is:

$$\max_{\{u_s(q)\}_{q,s,e}} \sum_s \phi_e^s \int \{q - \exp[u_s(q)]\} \frac{1}{\sigma_s}\varphi\left(\frac{q - h_s(e)}{\sigma_s}\right) dq$$

subject to (65), (66), and

$$\sum_s \phi_e^s \int u_s(q) \frac{1}{\sigma_s}\varphi\left(\frac{q - h_s(e)}{\sigma_s}\right) dq - C(e) \geq 0. \tag{67}$$

As in Grossman and Hart (1983), we break down this program in two parts. First, we consider the solution of the relaxed program holding each effort level $e \in [0, \bar{e}]$ fixed:

$$\min_{u(\cdot)} \sum_s \phi_e^s \int \exp[u_s(q)] \frac{1}{\sigma_s}\varphi\left(\frac{q - h_s(e)}{\sigma_s}\right) dq$$

subject to (65), (66), and (67).

The optimal contract to implement the lowest effort ($e^* = 0$) pays a fixed wage. The utility given to the manager is set at the lowest level that still satisfies both LL and IR: $u_s(q) = \max\{\ln\left(\bar{W}\right),\ C(0)\}$ for all q, s.

The next lemma obtains the solution of the relaxed program for $e^* > 0$.

Lemma 12 *The optimal contract that implements $e^* > 0$ in the relaxed program is:*

$$w_s(q) = \frac{\lambda}{\sigma_s^2} \cdot \max\{q - q_s^*, 0\},$$

where $q_s^* \leq \sigma_s^2 \frac{\bar{W}}{\lambda} + h_s(e^*)$ *(with equality if the IR does not bind).*

Proof. The Lagrangian associated with this program is:

$$\sum_s \phi_e^s \int \exp\left[u_s(q)\right] \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) dq +$$

$$\lambda \left[\sum_s \phi_e^s \int u_s(q) \frac{q - h_s(e^*)}{\sigma_s^3} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) dq - C'^*)\right]$$

$$+\mu_{IR} \left[\sum_s \phi_e^s \int u_s(q) \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) dq - C(e^*)\right]$$

$$+\mu_{LL}(q,s) \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) \phi_e^s u_s(q).$$

The FOC is:

$$-\exp\left[u_s(q)\right] \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) \phi_e^s + \lambda \frac{q - h_s(e^*)}{\sigma_s^3} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) \phi_e^s$$

$$+\mu_{IR} \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) \phi_e^s + \mu_{LL}(q,s) \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) \phi_e^s = 0,$$

where λ is the multiplier associated with (65), and μ_{LL} and μ_{IR} are the multipliers associated with (66), and (67). Since the program corresponds to the minimization of a strictly convex function subject to linear constraints, the FOC above, along with the standard complementary slackness conditions and the constraints, are sufficient for an optimum. Substitute $\exp\left[u_s(q)\right] = W_s(q)$ and simplify the FOC above to obtain:

$$-W_s(q) + \lambda \frac{q - h_s(e^*)}{\sigma_s^2} + \mu_{IR} + \mu_{LL}(q,s) = 0.$$

By the complementary slackness condition, we must have $\mu_{IR} \geq 0$ (with $\mu_{IR} = 0$ if IR does not bind). Similarly, $\mu_{LL}(q) \geq 0$ with equality if $W_s(q) > \bar{W}$. Thus, for $W_s(q) > \bar{W}$, we must have

$$W_s(q) = \lambda \frac{q - h_s(e^*)}{\sigma_s^2} + \mu_{IR} > \bar{W},$$

which can be rearranged as

$$q > \sigma_s^2 \frac{\bar{W} - \mu_{IR}}{\lambda} + h_s(e^*) =: q_s^*.$$

For $W_s(q) = \bar{W}$, we must have

$$\mu_{LL}(q,s) = \bar{W} - \lambda \frac{q - h_s(e^*)}{\sigma_s^2} - \mu_{IR} \geq 0 \iff q \leq q_s^*.$$

Combining both, we obtain

$$W_s(q) = \max\left\{\lambda \frac{q - h_s(e^*)}{\sigma_s^2} + \mu_{IR}, \bar{W}\right\} = \bar{W} + \frac{\lambda}{\sigma_s^2} \cdot \max\{q - q_s^*, 0\}.$$

Thus,

$$w_s(q) = \frac{\lambda}{\sigma_s^2} \cdot \max\{q - q_s^*, 0\}.$$

Finally, notice that, since $\mu_{IR} \geq 0$,

$$q_s^* = \sigma_s^2 \frac{\bar{W} - \mu_{IR}}{\lambda} + h_s(e^*) \leq h_s(e^*) + \sigma_s^2 \frac{\bar{W}}{\lambda},$$

with equality if IR does not bind (in which case, we have $\mu_{IR} = 0$). ■

We now obtain an upper bound on λ:

Lemma 13 *Suppose $e^* > 0$ is the effort that solves the firm's relaxed program. Then the optimal contract is*

$$w_s(q) = \max\left\{\frac{\lambda}{\sigma_s^2}(q - q_s^*),\ 0\right\},$$

where $0 < \lambda < \frac{\sqrt{2\pi}\sum_s \phi_{e*}^s h_s(e^*)}{\sum_s \frac{\phi_{e*}^s}{\sigma_s^2}}$ and $q_s^* \leq h_s(e^*) + \sigma_s^2 \frac{\bar{W}}{\lambda}$.

Proof. From the previous lemma, we need to show that $\lambda \leq \frac{\sqrt{2\pi}\sum_s \phi_{e*}^s h_s(e^*)}{\sum_s \frac{\phi_{e*}^s}{\sigma_s^2}}$. Recall that the optimal way to implement effort $e^* > 0$ is to pay the option:

$$w_s(q) = \max\left\{\frac{\lambda}{\sigma_s^2}(q - q_s^*),\ 0\right\},$$

where $q_s^* \leq h_s(e^*) + \sigma_s^2 \frac{\bar{W}}{\lambda}$. Since the firm's profits are increasing in the strike price q_s^* (holding all other variables, including effort, constant), her profits are bounded above by the profits from offering the option with the highest strike price for each signal s ($\bar{q}_s = h_s(e^*) + \sigma_s^2 \frac{\bar{W}}{\lambda} \geq q_s^*$), which equal

$$\sum_s \phi^s_{e^*} h_s(e^*) - \sum_s \phi^s_{e^*} \left[\frac{\lambda}{\sigma_s^2} \int_{h_s(e^*)+\sigma_s^2 \frac{\bar{W}}{\lambda}}^{\infty} \left(q - h_s(e^*) - \sigma_s^2 \frac{\bar{W}}{\lambda} \right) \frac{1}{\sigma_s} \varphi \left(\frac{q - h_s(e^*)}{\sigma_s} \right) dq \right].$$

For each s, let $z := q - h_s(e^*) - \sigma_s^2 \frac{\bar{W}}{\lambda}$, so that $q = z + h_s(e^*) + \sigma_s^2 \frac{\bar{W}}{\lambda}$. Note that $q \geq h_s(e^*) + \sigma_s^2 \frac{\bar{W}}{\lambda}$ if and only if $z \geq 0$. Thus, we can rewrite this expression as

$$\sum_s \phi^s_{e^*} h_s(e^*) - \sum_s \phi^s_{e^*} \left[\frac{\lambda}{\sigma_s^2} \int_0^{\infty} z \frac{1}{\sigma_s} \varphi \left(\frac{z + \frac{\sigma_s^2}{\lambda} \bar{W}}{\sigma_s} \right) dz \right].$$

Moreover, since $\varphi(z)$ is decreasing in z for $z > 0$, it follows that, for any s,

$$\varphi \left(\frac{z + \sigma_s^2 \frac{\bar{W}}{\lambda}}{\sigma_s} \right) < \varphi \left(\frac{z}{\sigma_s} \right) \quad \forall z > 0.$$

Thus, the firm's profits are strictly less than

$$\sum_s \phi^s_{e^*} h_s(e^*) - \sum_s \phi^s_{e^*} \frac{\lambda}{\sigma_s^2} \int_0^{\infty} \frac{z}{\sigma_s} \varphi \left(\frac{z}{\sigma_s} \right) dz.$$

Apply the following change of variables $y = \frac{z}{\sigma_s}$ (so that $z = \sigma_s y$, $dz = \sigma_s dy$) to write

$$\int_0^{\infty} \frac{z}{\sigma_s} \varphi \left(\frac{z}{\sigma_s} \right) dz = \sigma_s \int_0^{\infty} y \varphi(y)\, dy.$$

Integration by parts gives

$$\int_0^{\infty} y\varphi(y)dy = [-\varphi(y)]_0^{\infty} = \varphi(0) = \frac{1}{\sqrt{2\pi}}.$$

Substituting in the formula from before, it follows that the firm's profits are strictly less than

$$\sum_s \phi^s_{e^*} h_s(e^*) - \sum_s \phi^s_{e^*} \frac{1}{\sqrt{2\pi}} \cdot \frac{\lambda}{\sigma_s}.$$

Since the firm can always obtain a profit of zero by paying zero wages and implementing zero effort, we must have

$$\sum_s \phi_{e^*}^s h_s\left(e^*\right) - \frac{\lambda}{\sqrt{2\pi}} \sum_s \frac{\phi_{e^*}^s}{\sigma_s} > 0 \iff \lambda < \frac{\sqrt{2\pi}\sum_s \phi_{e^*}^s h_s\left(e^*\right)}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s}}.$$

■

The following lemma provides an additional upper bound:

Lemma 14 *For any $q_s^* \in \mathbb{R}\,\forall s$, $e \in [0,\bar{e}]$, $e^* \in [0,\bar{e}]$, $\sigma_s > 0\,\forall s$, and $\lambda > 0$, we have*

$$\begin{aligned}&\textstyle\sum_s \phi_e^s \int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot \max\left\{q - q_s^*,\ 0\right\}\right] (h_s'^2 \left[\frac{(q-h_s(e))^2}{\sigma_s^4} - \frac{1}{\sigma_s^2}\right] \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right) dq \\ &\leq \textstyle\sum_s \phi_e^s \int \left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot \max\left\{q - q_s^*,\ 0\right\}\right] (h_s'^2 \left[\frac{(q-h_s(e))^2}{\sigma_s^4} - \frac{1}{\sigma_s^2}\right] \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right) dq.\end{aligned}$$

Proof. For notational simplicity, for each s, let $y_s := \frac{q_s^* - h_s(e)}{\sigma_s}$, apply the change of variables $z_s := \frac{q-h_s(e)}{\sigma_s}$, and let

$$g_s(z) := \bar{W} + \frac{\lambda}{\sigma_s}\cdot \max\left\{z - y_s,\ 0\right\} - \ln\left[\bar{W} + \frac{\lambda}{\sigma_s}\cdot \max\left\{z - y_s,\ 0\right\}\right].$$

Then, the inequality in the lemma can be written as

$$\sum_s \frac{\phi_e^s}{\sigma_s^3}(h_s'^2 \int_{-\infty}^{\infty} g_s(z)\left(z^2-1\right)\varphi\left(z\right)dz \geq 0. \tag{68}$$

The terms ϕ_e^s, σ_s, and $(h_s'^2$ are positive, so it remains to prove that this integral is positive. We claim that, for each s, $g_s(\cdot)$ is non-decreasing. To see this, notice that, for $z_s \leq y_s$, $g_s(z) = \bar{W} - \ln \bar{W}$ (which is constant in z_s). For $z_s > y_s$, we have

$$g_s'(z) = \frac{\lambda}{\sigma_s}\left(\frac{\bar{W} - 1 + \frac{\lambda}{\sigma_s}\left(z-y\right)}{\bar{W} + \frac{\lambda}{\sigma_s}\left(z-y\right)}\right),$$

which is positive for all $z_s > y_s$ since $\bar{W} \geq 1$. Because g is non-decreasing, we have $g_s(q) \geq g_s(-q)$ for $q \geq 0$ and $\frac{d}{dq}\left[g_s(q) - g_s(-q)\right] \geq 0$. Note that, applying the change of variables $\tilde{z} = -z$ and using the symmetry of $\left(z^2-1\right)\varphi\left(z\right)$ around zero, we have:

$$\int_{-\infty}^{0} g_s(z)\left(z^2-1\right)\varphi\left(z\right)dz = -\int_{0}^{\infty} g_s(-z)\left(z^2-1\right)\varphi\left(z\right)dz. \tag{69}$$

Therefore,

$$\begin{aligned}\int g_s(z)\left(z^2-1\right)\varphi(z)\,dz &= \int_{-\infty}^{0} g_s(z)\left(z^2-1\right)\varphi(z)\,dz+\int_0^{\infty} g_s(z)\left(z^2-1\right)\varphi(z)\,dz\\ &= -\int_0^{\infty} g_s(-z)\left(z^2-1\right)\varphi(z)\,dz+\int_0^{\infty} g_s(z)\left(z^2-1\right)\varphi(z)\,dz\\ &= \int_0^{\infty}\left[g_s(z)-g_s(-z)\right]\left(z^2-1\right)\varphi(z)\,dz\\ &= \int_0^{1}\left[g_s(z)-g_s(-z)\right]\left(z^2-1\right)\varphi(z)\,dz+\int_1^{\infty}\left[g_s(z)-g_s(-z)\right]\left(z^2-1\right)\varphi(z)\,dz\\ &\geq \int_0^{1}\left[g_s(1)-g_s(-1)\right]\left(z^2-1\right)\varphi(z)\,dz+\int_1^{\infty}\left[g_s(1)-g_s(-1)\right]\left(z^2-1\right)\varphi(z)\,dz\\ &= \left[g_s(1)-g_s(-1)\right]\int_0^{\infty}\left(z^2-1\right)\varphi(z)\,dz=0,\end{aligned}$$

where the first line opens the integral between positive and negative values of z, the second line substitutes (69), the third line combines the terms from the two integrals, and the fourth line opens the integral between $z \leq 1$ and $z \geq 1$. The fifth line is the crucial step, which uses the following two facts: (i) $z^2 > (<)1$ for $z > (<)1$, and (ii) $g_s(z)-g_s(-z)$ is non-decreasing for all z. Therefore, substituting $g_s(z)-g_s(-z)$ by its upper bound where the term inside the integral is negative and by its lower bound where it is positive lowers the value of the integrand. The sixth line then combines terms and uses the fact that

$$\int_0^{\infty}\left(z^2-1\right)\varphi(z)\,dz=\left[-z\varphi(z)\right]_0^{\infty}=0.$$

■

The final lemma shows that the solution of the relaxed program also solves the firm's program if the effort cost is sufficiently convex (i.e., the FOA is valid).

Lemma 15 *Suppose* $C''(e) \geq \sum_s \phi_{e^*}^s h_s(\bar{e}) \frac{\sum_s \frac{\phi_e^s}{\sigma_s^3}(h_s'(e))^2}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s}}$ *for all* $e \in [0,\bar{e}]$*. Then, the solution of the firm's program coincides with the solution of the relaxed program.*

Proof. The manager's utility from choosing any effort $e \in [0,\bar{e}]$ is:

$$U(e;\{q_s^*\},\lambda) := \sum_s \phi_e^s \int \ln\left[\bar{W}+\frac{\lambda}{\sigma_s^2}\cdot\max\left\{q-q_s^*,\ 0\right\}\right]\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq-C(e).$$

We know from previous results that $0<\lambda<\frac{\sqrt{2\pi}\sum_s\phi_{e^*}^s h_s(\bar{e})}{\sum_s\frac{\phi_{e^*}^s}{\sigma_s}}$. The FOA is justified if

$$\frac{\partial^2 U}{\partial e^2}(e;\{q_s^*\},\lambda)\leq 0$$

for all $e \in [0, \bar{e}]$, all $q_s^* \in \mathbb{R}$, and $\lambda \in \left(0, \frac{\sqrt{2\pi}\sum_s \phi_{e^*}^s h_s(\bar{e})}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s}}\right)$. Differentiation gives

$$\begin{aligned}
\frac{\partial^2 U}{\partial e^2} &= \sum_s \int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q - q_s^*,\ 0\}\right]\frac{1}{\sigma_s}\frac{d^2}{de^2}\left[\phi_e^s \varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)\right]dq - C''(e) \\
&= \sum_s \int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q - q_s^*,\ 0\}\right]\frac{1}{\sigma_s}\frac{d}{de}\left[\frac{d\phi_e^s}{de}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right) + \phi_e^s\frac{d}{de}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)\right]dq - C''(e) \\
&= \sum_s \int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q - q_s^*,\ 0\}\right]\frac{1}{\sigma_s}\left[\frac{d^2\phi_e^s}{de^2}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right) + 2\frac{d\phi_e^s}{de}\frac{d}{de}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)\right. \\
&\quad \left. +\phi_e^s\frac{d^2}{de^2}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)\right]dq - C''(e) \\
&= \sum_s \frac{d^2\phi_e^s}{de^2}\int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q - q_s^*,\ 0\}\right]\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq \\
&\quad +2\sum_s \frac{d\phi_e^s}{de}\frac{h_s'(e)}{\sigma_s^2}\int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q - q_s^*,\ 0\}\right]\frac{q-h_s(e)}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq \\
&\quad +\sum_s \frac{\phi_e^s}{\sigma_s^2}\int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q - q_s^*,\ 0\}\right]\frac{1}{\sigma_s} \\
&\quad \times\left[(h_s'^2\left[\frac{(q-h_s(e))^2}{\sigma_s^2} - 1\right] + h_s''(e)\left(q - h_s(e)\right)\right]\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq - C''(e)
\end{aligned}, \tag{70}$$

where the last equality uses the fact that

$$\frac{d^2}{de^2}\left[\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)\right] = \frac{1}{\sigma_s^2}\left[(h_s'^2\left[\frac{(q-h_s(e))^2}{\sigma_s^2} - 1\right] + h_s''(e)\left(q - h_s(e)\right)\right]\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right).$$

First, with ϕ_e^s linear in e (assumption (ii)), $\frac{d^2\phi_e^s}{de^2} = 0\,\forall s$, so that the first term on the RHS of (70) is zero.

Second, the second term on the RHS of (70) can be rewritten as:

$$\begin{aligned}
&2\sum_s \frac{d\phi_e^s}{de}\frac{h_s'(e)}{\sigma_s^2}\int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q - q_s^*,\ 0\}\right]\frac{q-h_s(e)}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq \\
&\quad = 2\sum_s \frac{d\phi_e^s}{de}\frac{h_s'(e)}{\sigma_s^2}\left[\int_{-\infty}^{q_s^*}\ln\left(\bar{W}\right)\frac{q-h_s(e)}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq\right. \\
&\quad \left. + \int_{q_s^*}^{\infty}\ln\left(\bar{W} + \frac{\lambda}{\sigma_s^2}\left(q - q_s^*\right)\right)\frac{q-h_s(e)}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq\right],
\end{aligned} \tag{71}$$

where $q_s^* = \sigma_s^2 \frac{\bar{W}-\mu_{IR}}{\lambda} + h_s(e^*)$. For a given e, letting $\zeta_s := \frac{q-h_s(e)}{\sigma_s}$ and $\zeta_s^* := \frac{q_s^*-h_s(e)}{\sigma_s}$, we have:

$$\begin{aligned}
& \int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q-q_s^*,\ 0\}\right]\frac{q-h_s(e)}{\sigma_s}\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq \\
= & \int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s}\cdot\max\{\zeta-\zeta_s^*,\ 0\}\right]\zeta\varphi(\zeta)\,d\zeta \\
= & \int_{-\infty}^{\zeta_s^*} \ln\left[\bar{W}\right]\zeta\varphi(\zeta)\,d\zeta + \int_{\zeta_s^*}^{\infty}\ln\left[\bar{W} + \frac{\lambda}{\sigma_s}(\zeta-\zeta_s^*)\right]\zeta\varphi(\zeta)\,d\zeta \geq 0,
\end{aligned} \tag{72}$$

where the inequality follows from $\bar{W} \geq 1$ and the symmetry of the normal distribution. This shows that, in equation (71), the term in brackets is increasing in $h_s(e)$ and in $h_s'(e)$, and decreasing in σ_s, all else equal. Note that, as $\sum_s \phi_e^s = 1\,\forall e$, we have $\sum_s \frac{d\phi_e^s}{de} = 0$, which implies

$$\sum_{s|\frac{d\phi_e^s}{de}>0} \frac{d\phi_e^s}{de} = -\sum_{s|\frac{d\phi_e^s}{de}<0}\frac{d\phi_e^s}{de}.$$

In sum, with assumption (iii), the expression in (71) is negative.

Third, we show that the third term on the RHS of (70) is negative. Therefore, with $\phi_e^s \geq 0$ and $\sigma_s > 0$ for all s, with $h_s''(e) \leq 0$ for all s (assumption (i)), and with equation (72), we have:

$$\sum_s \frac{\phi_e^s}{\sigma_s}h_s''(e)\int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q-q_s^*,\ 0\}\right]\frac{q-h_s(e)}{\sigma_s}\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq \leq 0, \tag{73}$$

But notice that

$$\begin{aligned}
& \textstyle\sum_s \frac{\phi_e^s}{\sigma_s^2}\int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q-q_s^*,\ 0\}\right]\left[(h_s'^2\left[\frac{(q-h_s(e))^2}{\sigma_s^2}-1\right] + h_s''(e)(q-h_s(e))\right]\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq \\
= & \textstyle\sum_s \frac{\phi_e^s}{\sigma_s^2}\int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q-q_s^*,\ 0\}\right](h_s'^2\left[\frac{(q-h_s(e))^2}{\sigma_s^2}-1\right]\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq \\
& \textstyle+\sum_s \frac{\phi_e^s}{\sigma_s}h_s''(e)\int \ln\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q-q_s^*,\ 0\}\right]\frac{q-h_s(e)}{\sigma_s}\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq \\
\leq & \textstyle\sum_s \frac{\phi_e^s}{\sigma_s^2}(h_s'^2\int\left[\bar{W} + \frac{\lambda}{\sigma_s^2}\cdot\max\{q-q_s^*,\ 0\}\right]\left[\frac{(q-h_s(e))^2}{\sigma_s^2}-1\right]\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq \\
= & \textstyle\sum_s \lambda\frac{\phi_e^s}{\sigma_s^4}(h_s'^2\cdot\int\max\{q-q_s^*,\ 0\}\left[\frac{(q-h_s(e))^2}{\sigma_s^2}-1\right]\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq \\
= & \textstyle\sum_s \lambda\frac{\phi_e^s}{\sigma_s^4}(h_s'^2\cdot\int_{q_s^*}^{\infty}(q-q_s^*)\left[\frac{(q-h_s(e))^2}{\sigma_s^2}-1\right]\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq,
\end{aligned}$$

where the first equality separates the sum into two components, the inequality that follows uses the result from the previous lemma and equation (73), the next equality follows from the fact that $\int\left[\left(\frac{q-h_s(e)}{\sigma_s}\right)^2-1\right]\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq = 0$ (a Standard Normal variable has variance 1), and the last equality opens the max operator. Substituting in the expression from (70), we obtain the following sufficient condition for the validity of FOA:

$$\lambda \sum_s \frac{\phi_e^s}{\sigma_s^4} \left(h_s'(e)\right)^2 \cdot \int_{q_s^*}^{\infty} (q - q_s^*) \left[\left(\frac{q - h_s(e)}{\sigma_s} \right)^2 - 1 \right] \frac{1}{\sigma_s} \varphi \left(\frac{q - h_s(e)}{\sigma_s} \right) dq \leq C''(e) \tag{74}$$

for all $e \in [0, \bar{e}]$, $q_s^* \in \mathbb{R}$, and $\lambda \in \left(0, \frac{\sqrt{2\pi} \sum_s \phi_{e^*}^s h_s(\bar{e})}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s}}\right)$. Let

$$\xi_s(q_s^*) := \int_{q_s^*}^{\infty} (q - q_s^*) \left[\left(\frac{q - h_s(e)}{\sigma_s} \right)^2 - 1 \right] \frac{1}{\sigma_s} \varphi \left(\frac{q - h_s(e)}{\sigma_s} \right) dq.$$

We claim that

$$\xi_s'(q_s^*) \left\{ \begin{array}{c} > \\ < \end{array} \right\} 0 \iff q_s^* \left\{ \begin{array}{c} < \\ > \end{array} \right\} h_s(e). \tag{75}$$

Differentiation, gives:

$$\xi_s'(q_s^*) = - \int_{q_s^*}^{\infty} \left[\left(\frac{q - h_s(e)}{\sigma_s} \right)^2 - 1 \right] \frac{1}{\sigma_s} \varphi \left(\frac{q - h_s(e)}{\sigma_s} \right) dq. \tag{76}$$

But note that

$$\begin{aligned} \frac{d}{dq} \left[- \left(\frac{q - h_s(e)}{\sigma_s} \right) \varphi \left(\frac{q - h_s(e)}{\sigma_s} \right) \right] &= -\frac{1}{\sigma_s} \varphi \left(\frac{q - h_s(e)}{\sigma_s} \right) + \left(\frac{q - h_s(e)}{\sigma_s} \right)^2 \frac{1}{\sigma_s} \varphi' \left(\frac{q - h_s(e)}{\sigma_s} \right) \\ &= \left[\left(\frac{q - h_s(e)}{\sigma_s} \right)^2 - 1 \right] \frac{1}{\sigma_s} \varphi \left(\frac{q - h_s(e)}{\sigma_s} \right), \end{aligned}$$

where the first equality uses the fact that $\varphi'(q) = -q\varphi(q)$. Therefore,

$$\int \left[\left(\frac{q - h_s(e)}{\sigma_s} \right)^2 - 1 \right] \frac{1}{\sigma_s} \varphi \left(\frac{q - h_s(e)}{\sigma_s} \right) dq = - \left(\frac{q - h_s(e)}{\sigma_s} \right) \varphi \left(\frac{q - h_s(e)}{\sigma_s} \right).$$

Substituting back in (76), gives

$$\xi_s'(q_s^*) = - \left(\frac{q_s^* - h_s(e)}{\sigma_s} \right) \varphi \left(\frac{q_s^* - h_s(e)}{\sigma_s} \right) \left\{ \begin{array}{c} > \\ < \end{array} \right\} 0 \iff q_s^* \left\{ \begin{array}{c} < \\ > \end{array} \right\} h_s(e).$$

Therefore, $\xi_s(\cdot)$ is maximized at $q_s^* = h_s(e)$, so that, by condition (74), it suffices to show that

$$\lambda \sum_s \frac{\phi_e^s}{\sigma_s^4} \left(h_s'(e)\right)^2 \cdot \xi_s(h_s(e)) \leq C''(e), \tag{77}$$

for all $e \in [0, \bar{e}]$ and $\lambda \in \left(0, \frac{\sqrt{2\pi} \sum_s \phi_{e^*}^s h_s(\bar{e})}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s}}\right)$. Evaluating ξ_s at $h_s(e)$, gives:

$$\xi_s(h_s(e)) = \int_{h_s(e)}^{\infty} \frac{q - h_s(e)}{\sigma_s} \left[\left(\frac{q - h_s(e)}{\sigma_s}\right)^2 - 1\right] \varphi\left(\frac{q - h_s(e)}{\sigma_s}\right) dq.$$

Performing the change of variables $z_s := \frac{q - h_s(e)}{\sigma_s}$, we obtain

$$\xi_s(h_s(e)) = \sigma_s \int_0^{\infty} z\left(z^2 - 1\right) \varphi(z)\, dz. \tag{78}$$

Integration by parts, gives

$$\int z\left(z^2 - 1\right) \varphi(z)\, dz = -z^2\varphi(z) + \int z\varphi(z)\, dz,$$

where we let $(z^2 - 1)\,\varphi(z)\, dz = dv$ so that $v = -z\varphi(z)$, and we let $u = z$, so that $du = dz$. Therefore

$$\int_0^{\infty} z\left(z^2 - 1\right) \varphi(z)\, dz = \int_0^{\infty} z\varphi(z)\, dz.$$

Using the fact that $\frac{d}{dz}\left[-\varphi(z)\right] = z\varphi(z)$, it follows that

$$\int_0^{\infty} z\left(z^2 - 1\right) \varphi(z)\, dz = \left[-\varphi(z)\right]_0^{+\infty} = \varphi(0) = \frac{1}{\sqrt{2\pi}}.$$

Substituting in (78), yields

$$\xi_s(h_s(e)) = \frac{\sigma_s}{\sqrt{2\pi}}.$$

Substituting in condition (77), we obtain the following sufficient condition:

$$\frac{\lambda}{\sqrt{2\pi}} \sum_s \frac{\phi_e^s}{\sigma_s^3} \left(h_s'(e)\right)^2 \leq C''(e),$$

which is true for all $e \in [0, \bar{e}]$ and all $\lambda \in \left(0, \frac{\sqrt{2\pi}\sum_s \phi^s_{e*} h_s(\bar{e})}{\sum_s \frac{\phi^s_{e*}}{\sigma_s}}\right)$ if

$$\sum_s \phi^s_{e^*} h_s\left(\bar{e}\right) \frac{\sum_s \frac{\phi^s_e}{\sigma^3_s}\left(h'_s\left(e\right)\right)^2}{\sum_s \frac{\phi^s_{e*}}{\sigma_s}} \leq C''(e) \;\; \forall e \in [0, \bar{e}].$$

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