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# A Reconsideration of the Jensen-Meckling Model of Outside Finance

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# A Reconsideration of the Jensen-Meckling Model of Outside Finance

## 1. INTRODUCTION

This paper takes a new look at the approach to outside finance that was developed by Jensen and Meckling (1976) almost twenty years ago. This approach is summarized by the following propositions:

- Outside finance involves "agency costs" as relations between financiers on the one hand and entrepreneurs or managers on the other hand are affected by moral hazard on the side of entrepreneurs and managers.
- Different forms of outside finance involve different types of moral hazard and therefore different types of agency costs. Whereas outside equity finance is mainly affected by moral hazard concerning the level of effort exerted by the entrepreneur, outside debt finance is mainly affected by moral hazard concerning the riskiness of the strategy chosen by the entrepreneur.
- The equilibrium capital structure of the firm is one that minimizes the sum of all agency costs. To the extent that monitoring and bonding activities may play a role, the equilibrium capital structure is chosen jointly with these activities so as to minimize the sum of all agency, monitoring, and bonding costs.

Beyond the narrow issue of how to explain capital structure, the work of Jensen and Meckling has initiated a general research program of explaining observed financial institutions and financing patterns in terms of optimal (n-th best) responses to problems of moral hazard and incomplete information in financing relations. This research program has shaped the entire subsequent literature; for a survey, see Harris and Raviv (1991).

In recent years the direction of research has somewhat shifted. Jensen and Meckling have been criticized for focussing exclusively on the incentive implications of the return patterns of different instruments of outside finance. The "incomplete-contracts" approach of, e.g., Aghion and Bolton (1989, 1992), or Hart (1992) instead focusses on the assignment of control rights to the holders of different securities in different contingencies. Even so this approach continues to follow the overall research program of explaining observed financing patterns as optimal solutions to contracting problems in situations involving moral hazard and incomplete information.

The present paper is more old-fashioned. Whereas the "incomplete-contracts" literature has tended to neglect the return patterns of different types of securities altogether,[1] I take another look at the original Jensen-Meckling problem of explaining capital structure in terms of the incentive implications of return patterns associated with different mixes of instruments for outside finance. I am not trying to suggest that control rights considerations are unimportant. However, the incentive implications of different return patterns are also of interest. Indeed an important task of the "incomplete-contracts" approach must be to explain why control rights assignments and return patterns tend to be linked the way they are in debt and equity instruments. When Dewatripont and Tirole (1994) address this question, they go back to the proposition of Jensen and Meckling that outside equity finance is more susceptible to moral hazard concerning effort choices and outside debt finance is more susceptible to moral hazard concerning risk choices.

Jensen and Meckling did not actually provide the encompassing analysis that their conclusions would seem to require. They provided piecemeal analyses of (i) the incentive effects of outside equity finance on effort choices and (ii) the incentive effects of outside debt finance on risk choices. They did not show how the different pieces of the puzzle would fit together in the presence of both sources of moral hazard. This is where the present paper steps in.

The paper develops an integrated model in which there is moral hazard with respect to *both*, effort and risk choices of the entrepreneur.[2] It turns out that there is a natural interdependence between these two sources of moral hazard. The overall moral hazard problem takes on an extra dimension if the entrepreneur is able to conceal a low effort choice behind a relatively high level of returns induced by a high risk choice. Separate, piecemeal analyses

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1 An important exception is Dewatripont and Tirole (1994).

2 A rudimentary form of this model has previously been used in another context by Bester and Hellwig (1987). The model here is rather more general and therefore more suitable for dealing with the issues addressed by Jensen and Meckling.

of the different types of moral hazard by themselves are therefore insufficient. In particular, it is inappropriate to derive the total agency cost associated with a given mix of debt and equity finance simply by taking the agency cost of moral hazard concerning risk choices induced by debt finance and the agency cost of moral hazard concerning effort choices induced by equity finance and adding them up. Since risk taking and effort taking are just two sides of the same coin, it makes no sense to talk about their agency costs separately.

It also makes no sense to talk separately about the agency costs induced by equity finance and the agency costs induced by debt finance. Outside equity finance and debt finance *jointly* determine (i) the total investment that can be financed and (ii) the overall incentive scheme that the entrepreneur or manager faces as he makes his effort-and-risk choices. One must therefore think comprehensively in terms of the overall incentive effects of a given financing package on the combination of effort and risk levels that are chosen.

Indeed it is not clear at all that an optimal incentive scheme for the given moral hazard problem takes a form that can be interpreted in terms of standard financial instruments. Standard packages of financial instruments induce incentive schemes with a special mathematical structure, making the entrepreneur's income a piecewise linear, continuous function of overall returns. Why should a scheme with such a structure be suitable for dealing with a given incentive problem? Standard incentive theory at least tends to come up with much more complicated, highly nonlinear, even discontinuous incentive schemes.[3]

The combined effort-and-risk choice problem is actually more complicated than most conventional incentive problems[4] in that the moral hazard variable has two dimensions. Certain standard tools such as the monotone likelihood ratio property are therefore not available. In contrast to pure effort choice problems, in the combined effort-and-risk choice problem the role of output or

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3 The exception is Holmström and Milgrom (1987). However, the most satisfactory result that they have involves a pure effort choice model; they are quite explicit that it cannot be extended to involve risk choices as well (1987, p.324).

4 The exception is again Holmström and Milgrom (1987).

overall return is ambiguous. A high realization of output may signal a high level of effort, but it may also signal a high level of risk. Given that the incentive scheme is to discourage *both,* low effort *and* excessive risk taking, it is then not clear whether the entrepreneur should be rewarded or penalized when the realization of output or overall return is high.

Following most of the literature on financial contracting, from Jensen and Meckling (1976) to Innes (1990) or Dionne and Viala (1992), the paper assumes that all parties, the entrepreneur or manager as well as the financiers, are *risk neutral.* Risk sharing considerations play no role; there is no conflict between risk sharing and incentive considerations. Incentive considerations arise because of limitations to the entrepreneur's ability to repay the financiers, more precisely, because in any state of the world the entrepreneur's ability to repay his financiers depends on the return realization in that state of the world which in turn depends on the entrepreneur's prior risk and effort choices.

Given the assumption of risk neutrality on all sides, the paper finds that in a certain sense the problem of moral hazard with respect to risk choices is more robust than the problem of moral hazard with respect to effort choices: Whereas second-best outcomes *always* involve excessive risk taking, the qualitative properties of second-best effort levels seem highly dependent on the underlying technology. This finding is rather in contrast to a literature which tends to assign more weight to effort problems than to risk choice problems, see, e.g., Jensen and Meckling (1976), Innes (1990), Dionne and Viala (1992). However, there is a sense in which excessive risk taking may be interpreted as an instance of undereffort.

As for the original program of "explaining" the mix of outside debt finance and outside equity finance of a firm in terms of the incentive implications of the return pattern of the retained inside equity for the entrepreneur, the paper does indeed show that under certain circumstances an optimal incentive scheme for the entrepreneur may take the shape that would be generated by issuing a suitable mix of debt and equity instruments. However, this result does not reflect any deeper properties of debt and equity finance. Because of risk neutrality, there is a certain arbitrariness about incentives, and optimal incentive schemes are usually not unique.

The plan of the paper is as follows. Section 2 presents the basic model and formulates the second-best contracting problem. Section 3 relates the model to the literature. Sections 4-6 develop the analysis. Section 4 presents the main assumptions on the data of the model and derives a few general results on the unattainability of first-best outcomes, the strict desirability of outside finance in the second-best setting, the applicability of a first-order approach under the given assumptions. Section 5 provides a fairly detailed characterization of second-best contracts under the additional assumption that the relevant output variable is not perturbed by any noise. Section 6 discusses to what extent the presence of noise in the output variable would affect the implementability of outcomes that would be second-best in the absence of noise. In Section 7, the paper concludes with a brief discussion of the robustness of the main results. All proofs are given in the Appendix.

## 2. A MODEL OF OUTSIDE FINANCE WITH DOUBLE MORAL HAZARD

Consider the following situation involving moral hazard with respect to risk and effort choices at the same time. An entrepreneur with initial assets $A \geq 0$ wants to raise external funds I-A so as to finance an overall investment I. Once the investment is made, he chooses an effort level $\ell$ from a set $\mathcal{L} \subset \mathbb{R}_+$ and a risk class X from a set $\mathcal{X} \subset \mathbb{R}_+$. Given I, $\ell$, and X, the project earns a gross return $\tilde{y}$, which satisfies:

$$(1) \qquad \tilde{y} = \begin{cases} \tilde{\theta}\ X\ f(I,\ell) & \text{with probability } p(X), \\ 0 & \text{with probability } 1-p(X); \end{cases}$$

here $\tilde{\theta}$ is a positive-valued random variable with a given distribution function F(.) and expected value $E\tilde{\theta} = \int \theta\ dF(\theta) = 1$, f(.,.) is a standard production function, and p(.) is a decreasing function indicating the project's success probability as a function of the chosen risk class. With probability (1-p(X)), the project of risk class X will fail altogether.

I assume that the functions F(.), f(.,.), and p(.) are common knowledge. I also assume that the investment I is observable and verifiable by all parties. However, the effort choice $\ell$ and the risk choice X are not observable by outside financiers. These choices cannot be stipulated in the finance contract without further ado; they will depend on the incentives that the finance contract provides to the entrepreneur.

Besides the investment I and the outside funding I-A, the finance contract must stipulate the division of the gross return $\tilde{y}$ between the entrepreneur and his financiers. I assume that the realizations of $\tilde{y}$ are observable and verifiable, so in principle both the entrepreneur's return $\tilde{w}$ and the financiers' return $\tilde{r} = \tilde{y} - \tilde{w}$ may be taken to be arbitrary functions of output, without any additional incentive considerations.

The entrepreneur as well as the financiers are taken to be risk neutral. Given I, $\ell$, X, and the division $\tilde{w}$, $\tilde{r} = \tilde{y} - \tilde{w}$ of returns, expected payoffs are specified as

(2) $$E\tilde{w} - \ell$$

for the entrepreneur and

(3) $$E[\tilde{y}-\tilde{w}] - (I-A)$$

for the financiers.

Given the assumption of risk neutrality, there is no loss of generality in assuming that $\tilde{w}$ is given by a deterministic function of $\tilde{y}$; since both parties worry only about expected values, any additional randomization in $\tilde{w}$ conditional on $\tilde{y}$ can be averaged out, so $E\tilde{w}$ in (2) and (3) takes the form $E\tilde{w} = Ew(\tilde{y})$. Given (1), the payoff expectations (2) and (3) can thus be rewritten as

(2*) $$U(I,\ell,X,w(.)) := p(X) \int w(\theta X f(I,\ell))dF(\theta) + (1-p(X))w(0) - \ell$$

for the entrepreneur and

(3*) $$V(I,\ell,X,w(.)) := p(X) \int \left[\theta X f(I,\ell) - w(\theta X f(I,\ell))\right] dF(\theta) - (1-p(X))w(0) - (I-A)$$

for the financiers.

If there is Bertrand competition among financiers, the overall contracting problem can be written as

(4) $$\max_{I,\ell,X,w(.)} U(I,\ell,X,w(.))$$

subject to:

(5) $$V(I,\ell,X,w(.)) \geq 0$$

and

(6) $$U(I,\ell,X,w(.)) \geq U(I,\ell',X',w(.)) \quad \text{for all } \ell'\in\mathcal{L} \text{ and } X'\in\mathcal{X}.$$

Here (6) is the incentive compatibility condition on the entrepreneur's effort-and-risk choice $(\ell,X)$. In (4), it is required that $I\in\mathbb{R}_+$, $\ell\in\mathcal{L}$, $X\in\mathcal{X}$, and $w(.) \in \mathcal{W}$, where $\mathcal{W}$ is a set of admissible incentive schemes. The specification of the admissible sets $\mathcal{L}$, $\mathcal{X}$, and $\mathcal{W}$ is discussed in detail in the following sections.

As problem (4) is formulated, the interdependence of effort and risk choices is not very clear. Therefore, I reformulate the problem, making use of the fact that the impact of $\ell$ and X on the conditional distribution of $\tilde{y}$ given the event of success is entirely determined by the indicator $\bar{y} = Xf(I,\ell)$. In particular, the impact of $\ell$ and X on the conditional expectation of the entrepreneur's return $\tilde{w} = w(\tilde{y})$ given the event of success is entirely determined by $\bar{y}$ as $\bar{w}(\bar{y}) := \int w(\theta\bar{y})dF(\theta)$. Since $E\tilde{\theta}$ has been normalized to equal one, i.e. since $\int \theta dF(\theta) = 1$, the indicator $\bar{y}$ may be identified with the conditional expectation of $\tilde{y}$ given the event of success. Problem (4) may thus be rewritten as

(4*) $$\max_{\substack{I,\ell,X,w(.),\\ \bar{y},\bar{w}(.)}} \quad [p(X)\bar{w}(\bar{y}) + (1-p(X))\bar{w}(0) - \ell]$$

subject to the constraints:

(7) $$\bar{y} = Xf(I,\ell)$$

(5*) $$p(X)[\bar{y}-\bar{w}(\bar{y})] - (1-p(X))\bar{w}(0) - (I-A) \geq 0,$$

(6*) $$p(X)\bar{w}(\bar{y}) + (1-p(X))\bar{w}(0) - \ell \geq p(X')\bar{w}(\bar{y}') + (1-p(X'))\bar{w}(0) - \ell'$$
$$\text{for all } \ell'\in\mathcal{L},\ X'\in\mathcal{X}, \text{ and } \bar{y}' = X'f(I,\ell'),$$

(8) $$\bar{w}(\bar{y}') = \int w(\theta\bar{y}')dF(\theta) \quad \text{for all } \bar{y}' \geq 0.$$

A closer look at the incentive constraint (6*) shows that the overall incentive problem can be decomposed into two subproblems:

(i) How can one ensure that a given value $\bar{y}$ of the return indicator is achieved by the proper effort-and-risk choice? In particular, how can the entrepreneur be motivated so that he doesn't try to achieve the same $\bar{y}$ with less effort and more risk?

(ii) How can the entrepreneur be induced to aim for the optimal value $\bar{y}$ of the return indicator?

Problem (i) highlights the interdependence of the two sources of moral hazard in the present context. Whereas the entrepreneur's risk choice is usually analysed in terms of the tradeoff between the probability of success and his return in the event of success, here it involves a tradeoff between the probability of success and the effort required to achieve the expected return $\bar{y}$ in the event of success. A given $\bar{y}$ can be achieved through a low-effort/high-risk strategy as well as a high-effort/low-risk strategy. Financiers would prefer the latter; however the entrepreneur is unwilling to comply unless the difference $\bar{w}(\bar{y})-\bar{w}(0)$ between his conditionally expected returns in the events of success and of failure is sufficiently large.

To put the matter formally, define

$$U^*(\bar{y}',\bar{w}',I) := \sup_{\ell'\in\mathcal{L}} \left[p\left(\frac{\bar{y}'}{f(I,\ell')}\right) \bar{w}' - \ell'\right] \qquad (9)$$

for any $\bar{y}'$, $\bar{w}'$, I. The incentive compatibility condition (6*) can then be rewritten as a pair of constraints:

$$p(X)[\bar{w}(\bar{y})-\bar{w}(0)] - \ell = U^*(\bar{y},\ \bar{w}(\bar{y}) - \bar{w}(0),I) \qquad (6^*a)$$

and

$$U^*(\bar{y},\ \bar{w}(\bar{y})-\bar{w}(0),I) \geq U^*(\bar{y}',\ \bar{w}(\bar{y}')-\bar{w}(0),I) \quad \text{for all } \bar{y}'\geq 0. \qquad (6^*b)$$

Condition (6*a) requires simply that for $(\bar{y}',\bar{w}',I) = (\bar{y},\bar{w}(\bar{y}) - \bar{w}(0),I)$ the supremum in (9) be attained at $\ell$.

Notice that condition (6*a) hinges *only* on the difference $\bar{w}(\bar{y}) - \bar{w}(0)$. In contrast, condition (6*b) hinges on the comparison of $\bar{w}(\bar{y}) - \bar{w}(0)$ with $\bar{w}(\bar{y}')$ $- \bar{w}(0)$ for any $\bar{y}'$. Here the disturbance term $\tilde{\theta}$ may play a role. If there is no disturbance, i.e., if $\tilde{\theta} \equiv 1$,(6*b) can fulfilled by setting $w(\bar{y}') = w(0)$ for $\bar{y}'$ $\neq \bar{y}$; this implies $\bar{w}(\bar{y}') = \bar{w}(0)$ and $U^*(\bar{y}',0,I) = 0$ for $\bar{y}' \neq \bar{y}$, ensuring that

(6b*) holds. If $\tilde{\theta}$ is a nondegenerate random variable, this device may be unavailable; for instance, if the distribution of $\tilde{\theta}$ has a continuous density, (8) implies that $\bar{w}(.)$ is a continuous function so for $\bar{y}'$ close to $\bar{y}$, $U^*(\bar{y}', \bar{w}(\bar{y}')-\bar{w}(0), I)$ must be positive if $U^*(\bar{y}, \bar{w}(\bar{y})-\bar{w}(0), I)$ is.

## 3. RELATION TO THE LITERATURE

Before I proceed with the analysis, I briefly discuss the relation of the model presented here to the literature. Several previous contributions can be seen as special cases of the model formulated above, each one involving different assumptions about the functions F(.), f(.,.), and p(.) as well as

- the set $\mathcal{L}$ of available effort choices,
- the set $\mathcal{X}$ of available risk choices, and
- the set $\mathcal{W}$ of admissible incentive schemes.

Up to now I have been deliberately vague about the sets $\mathcal{L}$, $\mathcal{X}$, and $\mathcal{W}$. This literature review seems the best place to discuss them in detail.

In term of the present model, the analysis of *effort choice and equity finance in Jensen and Meckling (1976)* corresponds to the case $\tilde{\theta} \equiv 1$ (no disturbance in returns), $\mathcal{L} = \mathbb{R}_+$, f concave and $\mathcal{X} = \{1\}$, $p(1) = 1$ (no risk choice); there is no return uncertainty at all. Under pure equity finance, $\mathcal{W}$ is restricted to the set of functions w(.) taking the form $w(y) = (1-\alpha)y$ where $\alpha \in [0,1]$ is the share of outside equity in the firm. Problem (4*) then takes the form:[5]

$$\max_{I,\ell,\alpha} \ (1-\alpha)f(I,\ell)-\ell$$

$$\text{subject to:} \quad \alpha f(I,\ell) - (I-A) \geq 0$$

$$\text{and:} \quad (1-\alpha)f_2(I,\ell) = 1.$$

Jensen and Meckling assume that the entrepreneur's initial wealth A is less than the investment I* that is required for the first-best outcome $(I^*,\ell^*)$ satisfying $f_1(I^*,\ell^*) = f_2(I^*,\ell^*) = 1$. Under pure equity finance then the

5 Since f has been assumed to be concave, for given $\alpha$ and I, the effort choice problem $\text{Max}[(1-\alpha)f(I,\ell)-\ell]$ has a unique solution, and the incentive constraint (6*) may be replaced by the corresponding first-order condition.

first-best outcome cannot be achieved: If $A<I^*$, one has either $I<I^*$ or $I>A$, with $\alpha>0$ and $f_2(I,\ell) > (1-\alpha)f_2(I,\ell) = 1$.

Jensen and Meckling themselves observe that the given agency problem can be solved without agency costs if pure debt finance is admitted. Under pure debt finance, $\mathcal{W}$ is the set of functions w(.) taking the form $w(y) = \max(0,y-R)$ where R is the firm's obligation to its creditors. In this case the first-best outcome $(I^*,\ell^*)$ is achieved by setting $I=I^*$, $\ell=\ell^*$, $R=I^*-A$.

From the perspective of abstract incentive theory, it is not clear whether this latter result should really be interpreted as a result about debt finance. If one looks at the problem in terms of the decomposition into subproblems (i) and (ii) in Section 2, one finds that (i) in the absence of a risk choice problem there is only one effort level $\ell$ that corresponds to a given target return $\bar{y} = f(I,\ell)$, and (ii) in the absence of a return disturbance the realized return y is just equal to the target return $\bar{y}$: (i) means that the problem of setting incentives for effort choices is equivalent to the problem of setting incentives for the target return $\bar{y}$; (ii) means that incentives for a target return $\bar{y}$ can be directly incorporated in the incentive scheme w(.). The first-best outcome $(I^*,\ell^*)$ with return $\bar{y}=f(I^*,\ell^*)$ can thus be implemented simply by setting

$$w(y) = \begin{cases} f(I^*,\ell^*) - (I^*-A), & \text{if } y = f(I^*,\ell^*). \\ 0 & \text{otherwise.} \end{cases}$$

Further insight about this issue is provided by Innes (1990). In terms of the present model, his analysis corresponds to the case where $\tilde{\theta}$ is a nondegenerate random variable with range $\mathbb{R}_{++}$, but otherwise the assumptions $\mathcal{L} = \mathbb{R}_+$, $\mathcal{X} = \{1\}$, $p(1)=1$ of the effort choice model of Jensen and Meckling are retained. Innes restricts $\mathcal{W}$ to be the set of functions w(.) such that both w(y) and $r(y) = y-w(y)$ are nonnegative and r(y) is nondecreasing in y. Given this restriction, he confirms the finding of Jensen and Meckling that pure debt finance is optimal when there is only moral hazard with respect to effort choices, i.e., he finds that a solution to problem (4*) necessarily involves an incentive scheme of the form $w(y) = \max(0,y-R)$. Attainment of the first-best outcome $(I^*,\ell^*)$ is precluded by the disturbance $\tilde{\theta}$ and the nonnegativity and monotonicity conditions.[6] Even so, it is desirable to have a contract penalizing the entrepreneur for low return realizations and rewarding him for high return realizations. Within the class $\mathcal{W}$ that Innes considers, the incentive schemes w(.) that correspond to pure debt finance do this most effectively.

In terms of the present model, the analysis of *risk choice and debt finance in Jensen and Meckling (1976)* corresponds to the specification $\mathcal{L} = \{1\}$ (no effort choice), and, e.g., $\mathcal{X} = [\underline{X},\infty)$, with p(.) a decreasing function of $\mathcal{X}$ into [0,1]. Under pure debt finance, $\mathcal{W}$ is restricted to the set of functions w(.) taking the form w(y) = max (0,y-R), so problem (4*) takes the form:

$$\max_{I,X,R} \; p(X) \int_{R/Xf(I,1)} [\theta X f(I,1) - R] dF(\theta) - 1$$

subject to:
$$p(X)Xf(I,1) \int_0^{R/Xf(I,1)} \theta \, dF(\theta) + p(X)R(1-F(R/Xf(I,1))) - (I-A) \geq 0$$

and
$$p(X) \int_{R/Xf(I,1)}^{\infty} [\theta X f(I,1)-R] \, dF(\theta) \geq$$

$$p(X') \int_{R/X'f(I,1)}^{\infty} [\theta X' f(I,1)-R] dF(\theta) \text{ for all } X' \in \mathcal{X}.$$

A first-best outcome $(I^*,X^*)$ is now given by the conditions $f_1(I^*,1) = 1$ and $X^* \in \arg\max p(X)X$. Under pure debt finance this outcome cannot be achieved if $A<I^*$. In this case $I=I^*$ would require $R>0$ and the incentive-compatibility constraint would preclude $X=X^*$. For instance if $\tilde{\theta} \equiv 1$, the incentive-compatibility constraint requires $X \in \arg\max p(X')[X'f(I,1) - R]$, which implies $X>X^*$ as the entrepreneur neglects the fact that an increase in X lowers the expected value p(X)R of the financiers' nonbankruptcy return. More generally, the entrepreneur's return $\tilde{w}$ under pure debt finance is given by a *convex* function of $\tilde{y}$, so that at $X^* \in \arg\max p(X)X$, he is willing to raise X and reduce the expected value of $\tilde{y}$ in order to obtain a higher variance of $\tilde{y}$; this imposes a negative externality on the financiers, whose return $\tilde{r} = \tilde{y}-\tilde{w}$

6 Without these conditions on w(.) and r(.) attainability of the first-best would follow from risk neutrality, see Grossman and Hart (1983).

is given by a concave function of $\tilde{y}$, so they suffer from both, the reduction in the expected value and the increase in the variance of $\tilde{y}$.[7]

Jensen and Meckling also note that the risk choice agency problem is solved without agency costs if pure equity finance is used. In this case the first-best outcome $(I^*,X^*)$ is achieved by setting $I=I^*$, $X=X^*$, $\alpha = (I^*-A)/p(X^*)X^*f(I^*,1)$ and noting that the incentive-compatibility condition for X requires $X \in \arg\max p(X')X'$. Under pure equity finance, the entrepreneur's returns are proportional to $\tilde{y}$ so X is chosen so as to make $E\tilde{y}$ maximal; there is no conflict of interest about risk choice.[8]

In the case of moral hazard with respect to risk choices, the attainability of a first-best outcome by equity finance is precluded if one introduces *another* information problem. Thus suppose that financiers cannot observe the realization of $\tilde{y}$ unless they spend resources for this purpose. Extending the arguments of Townsend (1979) and Gale and Hellwig (1985) one then finds that the optimal second-best arrangement involves pure debt finance with return verification in bankruptcy states, and the first-best outcome $(I^*,X^*)$ is not attained (Bester and Hellwig (1987)). However in that framework there is no room for outside equity. Therefore I shall not pursue this line of argument and instead retain the assumption that return realizations are costlessly observable and verifiable.

The separate analyses of moral hazard with respect to effort choices and moral hazard with respect to risk choices are based on different model specifications, i.e., different assumption about technologies. To study what happens when there is moral hazard with respect to *both*, effort choices and risk choices, at the same time, one needs again a different model, one in which $\mathcal{L}$ and $\mathcal{X}$ are *both* nondegenerate sets. Such a combined effort-and-risk choice problem has a different quality from either the effort choice problem or the risk choice problem separately. To see this, consider the case $\tilde{\theta}\equiv 1$ when the disturbance in returns plays no role. In this case, the first-best outcome

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7 A more extensive discussion of this problem is given by Stiglitz and Weiss (1981) in their analysis of equilibrium credit rationing; see also Bester and Hellwig (1987).

8 Remarkably, this conclusion is independent of what one assumes about $\tilde{\theta}$. Nondegeneracy of $\tilde{\theta}$ affects the efficiency of a forcing scheme of the form $w(y) = X^*f(I^*,1) - (I^*-A)$ if $y = X^*f(I^*,1)$, $w(y) = 0$ otherwise, but it does not affect the efficiency of the equity contract.

can be implemented in a pure effort choice problem because the return $\bar{y} = f(I^*,\ell)$ is a perfect signal of the effort level $\ell$, and low effort can be discouraged by penalizing the entrepreneur for low values of returns. In a pure risk choice problem, the first-best outcome can also be implemented; here the return $\bar{y} = Xf(I^*,1)$ in the event of success is a perfect signal of the risk class X, and excessive risk taking can be discouraged by penalizing the entrepreneur for high values of returns. In contrast, in a combined effort-and-risk choice problem, a first-best outcome cannot be implemented. Eeven if $\tilde{\theta} \equiv 1$ the entrepreneur's behaviour cannot be fully inferred from the return $\bar{y} = Xf(I^*,\ell)$ in the event of success. Therefore it is not enough to penalize him for unwanted values of returns; it is also necessary to provide him with the proper incentive to choose the desired combination of $\ell$ and X among those yielding the desired $\bar{y}$ in the event of success. As will be shown below, this requirement in conjunction with the participation constraint (5*) for the financiers precludes the attainment of a first-best outcome.

## 4. ANALYSIS OF THE MODEL: SOME GENERAL RESULTS

I study the contracting problem (4*) under the assumption that $\mathcal{L} = \mathbb{R}_+$ and $\mathcal{X} = \mathbb{R}_+$, i.e., that there is nontrivial moral hazard with respect to *both*, effort and risk choices. I impose the following assumptions:

A.1. The production function $f: \mathbb{R}^2_+ \to \mathbb{R}_+$ is continuous and nondecreasing. It is twice continuously differentiable, strictly increasing and strictly concave on $\mathbb{R}^2_{++}$, with $\lim_{I\to 0} f_1(I,\ell) = \infty$ for all $\ell>0$, $\lim_{\ell\to 0} f_2(I,\ell) = \infty$ for all $I>0$, and $\lim_{\delta\to 0} f_1(\delta I_0,\delta\ell_0) = \lim_{\delta\to 0} f_2(\delta I_0,\delta\ell_0) = \infty$, $\lim_{\delta\to\infty} f(\delta I_0,\delta\ell_0)/\delta = 0$ for some $I_0>0$, $\ell_0>0$.

A.2. The success probability function $p: \mathbb{R}_+ \to [0,1]$ is continuous and nonincreasing. There exists $\bar{X} \in \mathbb{R}_{++}$ such that (i) the function $X \to Xp(X)$ is twice continuously differentiable and strictly concave, with $p''(X)X + 2p'(X) < 0$, on $(0,\bar{X})$, and (ii) $p(X) = 0$ for all $X \geq \bar{X}$.

Assumption A.1 is fairly standard. Differentiability is not essential; it serves mainly to keep the presentation simple. The Inada conditions are imposed to rule out boundary solutions. As for A.2, the monotonicity of p(.)

corresponds to the notion of a tradeoff between the return parameter X and the success probability p(X). Differentiability again serves to simplify the exposition. The one condition that is not straightforward is the requirement that Xp(X) be concave on the set of all X for which p(X) is strictly positive. As will be seen below, this requirement is convenient because it permits a first-order approach to the incentive constraint (6*a). However, it rules out the possibility that p(X) is positive on all of $\mathbb{R}_+$. Thus for instance the specification $p(X) = e^{-X}$ is ruled out.

A vector $(I^*,\ell^*,X^*)$ of investment, effort, and risk choices is called a *first-best outcome* if it maximizes the overall expected surplus $p(X)Xf(I,\ell)-\ell-I$. By standard arguments, Assumptions A.1 and A.2 ensure the existence of a unique first-best outcome; moreover the first-best outcome is strictly positive and coincides with the unique solution to the first-order conditions

(10a) $\quad p(X^*)X^*f_1(I^*,\ell^*) - 1 = 0,$

(10b) $\quad p(X^*)X^*f_2(I^*,\ell^*) - 1 = 0,$

(10c) $\quad p(X^*) + p'(X^*)X^* = 0.$

If the entrepreneur's initial assets are sufficiently large, the first-best outcome can be implemented despite the unobservability of effort and risk choices by outsiders. One easily verifies that $I=I^*$, $\ell=\ell^*$, $X=X^*$, $w(.) = w^*(.)$ where $w^*(y) \equiv A - I^* + y$, is a solution to the contracting problem (4) whenever A is large enough for $w^*(.)$ to be an admissible incentive scheme. To eliminate this possibility, I impose the further assumptions:

A.3. $\quad \mathcal{W} = \{w(.) | w(y) \geq 0 \quad \text{for all } y\}$

A.4. $\quad A < I^*.$

A negative value of w(y) for some y would correspond to a payment r(y) to the financiers in excess of y. Assumption A.3 indicates that such a payment is infeasible because the entrepreneur has no additional source of funds to finance it. More precisely, any source of funds outside the firm is already included in the initial assets A, so once these assets are brought into the enterprise, there are no further funds to provide for payments from the entrepreneur to the financiers in excess of the firm's returns. Assumption A.4 then imposes the further condition that A is insufficient to finance the first-best investment $I^*$. Thus, A.3 and A.4 together imply that the incentive scheme $w^*(.)$ with $w^*(y) = A - I^* + y$ for all $y$ is inadmissible.

Indeed, *under A.3 and A.4, the first-best outcome cannot be implemented at all.* To see this, note that the incentive constraint (6*a) implies

$$(11) \qquad p'(X)X\left[\bar{w}(\bar{y}) - \bar{w}(0)\right] + \frac{f(I,\ell)}{f_2(I,\ell)} = 0$$

as the first-order condition for the function

$$\ell' \to p\left(\frac{\bar{y}}{f(I,\ell')}\right)[\bar{w}(\bar{y}) - \bar{w}(0)] - \ell'$$

to have a maximum at $\ell' = \ell$ with $X = \bar{y}/f(I,\ell)$. If the first-best outcome were implementable, (11) would have to hold for $I=I^*$, $\ell=\ell^*$, $X=X^*$, and $\bar{y} = X^*f(I^*,\ell^*)$. In view of (10b) and (10c), this would imply

$$-p(X^*)[\bar{w}(\bar{y}) - \bar{w}(0)] + p(X^*)X^*f(I^*,\ell^*) = 0$$

or:

$$\bar{w}(\bar{y}) = \bar{w}(0) + \bar{y}.$$

But then the participation constraint (5*) for the financiers would imply

$$-\bar{w}(0) - (I^*-A) \geq 0,$$

in contradiction to either A.3 or A.4.

Given Assumptions A.1–A.4, I study problem (4*) as a *second-best* incentive contracting problem. I begin with the observation that even though the first-best outcome cannot be achieved, it is desirable to have at least some outside finance.

PROPOSITION 4.1

Assume A.1–A.4. Any solution $(I,\ell,X,w(.),\bar{y},\bar{w}(.))$ to problem (4*) satisfies $I > A$.

The argument for Proposition 4.1 is quite different in the two cases $A>0$ and $A=0$. For $A>0$, the point of the argument is that the agency cost of the first unit of outside finance is zero so for $A<I^*$, the marginal benefit of an additional investment necessarily exceeds the sum of the marginal opportunity cost and the marginal agency cost. This is because with $f(A,\hat{\ell}) > 0$, a small increase in I beyond A requires only a small equity share for outside financiers, so the effect on effort is also small; moreover at I=A and $\ell=\hat{\ell}$, the social marginal benefits and costs of changing effort are equal, so the small change in effort has no first-order effects on expected total surplus.

In contrast, for A=0, Proposition 4.1 is based on the Inada conditions, i.e., the assumption that initially the marginal benefits of additional investment and effort are large, outweighing whatever the marginal agency costs may be. If $f(0,\hat{\ell}) = 0$, outside equity finance for even the first unit of investment requires the issue of a significant share of the venture, and one cannot rely on any argument about adverse incentive effects of this equity issue being of the second order of smalls. Indeed the specification $f(I,\ell) = c \ln(1+(I\ell)^{1/2})$ provides an example for the possibility that without the Inada condition imposed in A.1, outside finance of investment by a pure equity contract may not be possible at all even though $I^* > 0$ and $\ell^* > 0$. For this specification, one has $I^* = \ell^* = \max(0, c\, p(X^*)X^*/2 - 1)$, hence $I^*>0$ and $\ell^*>0$ if $c\, p(X^*)X^* > 2$; at the same time for A=0 and $c\, p(X^*)X^* \leq \sqrt{8}$, one finds that for I > 0 there is no effort level so that *both*, the financiers' participation constraint (5*) and the entrepreneur's incentive constraint (6*) are satisfied at the same time.

The Inada conditions in A.1 also yield:

PROPOSITION 4.2
Assume A.1–A.4. Any solution $(I,\ell,X,w(.),\bar{y},\bar{w}(.))$ to problem (4*) satisfies $\ell>0$.

Neither Proposition 4.1 nor 4.2 requires the full strength of Assumption A.2. These results only require that p(X)X has a strictly positive maximum $p(X^*)X^* > 0$ at some point $X^*$. Uniqueness of the maximum is not required, so neither the monotonicity of p(X) nor the curvature of p(X)X play any role for these results. The reason for this is that the argument for Propositions 4.1 and 4.2 involves only incentive schemes of the form $w(y) = w(0) + (1-\alpha)y$, which automatically induce the entrepreneur to choose a first-best risk level.

However incentive schemes of the form $w(y) = w(0) + (1-\alpha)y$ are *not* in general optimal. For other incentive schemes, the incentive constraint (6*) is difficult to handle. The full strength of Assumption A.2 is used in the following result, which provides a handle on (6*a) and thereby makes problem (4*) analytically tractable.

LEMMA 4.3

Assume A.1 and A.2. A contract $(I,\ell,X,w(.),\bar{y},\bar{w}(.))$ with $\ell>0$ and $\bar{y} = Xf(I,\ell)$ satisfies the incentive constraint (6*a) if and only if it satisfies the first-order condition

(11) $$\bar{w}(\bar{y}) - \bar{w}(0) = \frac{f(I,\ell)}{-p'(X)Xf_2(I,\ell)}$$

and moreover

(12) $$p(X)[\bar{w}(\bar{y}) - \bar{w}(0)] - \ell \geq 0.$$

Given Proposition 4.2 and Lemma 4.3, one can replace the incentive constraint (6*a) by conditions (11) and (12). Given (11), one can substitute for $[(\bar{w}(\bar{y}) - \bar{w}(0)]$ in (4*), (5*), (6*b), and (12). The contracting problem (4*) is then rewritten as:

(13) $$\max_{\substack{I,\ell,X,w(.)\\ \bar{y},\bar{w}(.)}} \left[r(X)\,\frac{f(I,\ell)}{f_2(I,\ell)}\right] - \ell + \bar{w}(0)$$

subject to the constraints

(14) $$p(X)Xf(I,\ell) - r(X)\,\frac{f(I,\ell)}{f_2(I,\ell)} - \bar{w}(0) \geq I - A,$$

(15) $$r(X)\,\frac{f(I,\ell)}{f_2(I,\ell)} - \ell \geq 0$$

(16) $$r(X)\,\frac{f(I,\ell)}{f_2(I,\ell)} - \ell \geq U^*(\bar{y}',\bar{w}(\bar{y}')-\bar{w}(0),I) \quad \text{for all } \bar{y}' \neq \bar{y}$$

as well as (7), (8), and (11), where

(17) $$r(X) := p(X)/(-p'(X)X)$$

I now impose the additional assumption that $\tilde{\theta} \equiv 1$, so that the observed return $\tilde{y}$ in the event of success is always equal to the target return $\bar{y}$. As discussed above, this assumption facilitates the analysis because it reduces (8) to the requirement that $\bar{w}(y) = w(y)$ for all y. This means that for $\bar{y}' \neq \bar{y}$ one can set $\bar{w}(\bar{y}') - \bar{w}(0) = 0$ regardless of the fact that for (11) to hold with $\ell > 0$ one must have $\bar{w}(\bar{y}) - \bar{w}(0) > 0$. Given $\bar{w}(\bar{y}') - \bar{w}(0) = 0$, no further attention needs to be paid to the incentive constraint (16) (resp. (6*b)) as it coincides with (15).

To solve the contracting problem (13), it then suffices to solve the Lagrangian problem:

$$(18) \qquad \max_{\substack{I,\ell,X,\\ \bar{w}(0)\geq 0}} \ \min_{\mu\geq 0,\nu\geq 0} \Bigg[ r(X)\frac{f(I,\ell)}{f_2(I,\ell)} - \ell + \bar{w}(0) + \mu\left(p(X)Xf(I,\ell) - r(X)\frac{f(I,\ell)}{f_2(I,\ell)} - I + A - \bar{w}(0)\right) + \nu\left(r(X)\frac{f(I,\ell)}{f_2(I,\ell)} - \ell\right)\Bigg].$$

Given a solution to this Lagrangian problem, the full solution to problem (13) is obtained by using (7) and (11) to determine $\bar{y}$ and $\bar{w}(\bar{y})$, by setting $\bar{w}(\bar{y}') = \bar{w}(0)$ for $\bar{y}' \neq \bar{y}$, and by using (8) to determine $w(.) = \bar{w}(.)$.

The analysis of problem (18) is simplified by the following lemmas.

LEMMA 5.1

Assume A.1-A.4. Any solution $(I,\ell,X,\bar{w}(0),\mu,\nu)$ to problem (18) satisfies $\mu > 1$ and $\bar{w}(0) = 0$.

LEMMA 5.2

Assume A.1-A.4. Any solution $(I,\ell,X,\bar{w}(0),\mu,\nu)$ to problem (18) satisfies $\nu=0$.

To understand these lemmas, note that the variable $\bar{w}(0)$ provides for the possibility of side payments from the financiers to the entrepreneur.

Because of this possibility, the multiplier $\mu$ on the financiers' participation constraint cannot be less than one. Moreover, it cannot be equal to one because $\mu$=1 would induce the first-best outcome $(I^*,\ell^*,X^*)$, which violates the financiers' participation constraint. However, with $\mu>1$, the optimal value of the "side payment" $\bar{w}(0)$ must be zero. With $\bar{w}(0) = 0$, the objective in (13) coincides with the left hand side of the incentive constraint (15), so this constraint cannot be strictly binding, and one must have $\nu=0$.

Given Lemmas 5.1 and 5.2, problem (18) can again be simplified and reformulated in the form:

$$(19)\qquad \underset{I,\ell,X}{\text{Max}}\ \underset{\mu\geq 1}{\text{Min}} \Big[p(X)Xf(I,\ell) - \ell - I + A + (\mu-1)\Big(p(X)Xf(I,\ell) - r(X)\frac{f(I,\ell)}{f_2(I,\ell)} - I + A\Big)\Big].$$

The first-order conditions for I, $\ell$, X and $\mu$ can be written as:

$$(20)\qquad p(X)Xf_1(I,\ell) - 1 = \frac{\mu-1}{\mu}\, r(X)\left[\frac{f_1(I,\ell)}{f_2(I,\ell)} - \frac{f(I,\ell)f_{21}(I,\ell)}{f_2(I,\ell)^2}\right]$$

$$(21)\qquad p(X)Xf_2(I,\ell) - 1 = \frac{\mu-1}{\mu}\, r(X)\left[1 - \frac{f(I,\ell)f_{22}(I,\ell)}{f_2(I,\ell)^2}\right] - \frac{\mu-1}{\mu}$$

$$(22)\qquad (p(X) + p'(X)X)f_2(I,\ell) = \frac{\mu-1}{\mu}\, r'(X)$$

$$(23)\qquad p(X)Xf(I,\ell) - r(X)\frac{f(I,\ell)}{f_2(I,\ell)} - I + A = 0$$

From (22) and (23), one obtains:

PROPOSITION 5.3

Assume A.1–A.4, and let $\tilde{\theta}\equiv 1$. Any solution $(I,\ell,X,w(.),\bar{y},\bar{w}(.))$ to problem (4*) satisfies $X > X^*$ and $r(X) < 1$.

*Any second-best outcome involves excessive risk taking, i.e., a risk choice X with success probability p(X) less than the first-best success probability p(X*).* The intuition for this result is the following: A change in X affects both, the term p(X)X in the expected return on the entrepreneur's venture and the term $r(X) = p(X)/(-p'(X)X)$ in the expected reward for success, $p(X)[\bar{w}(\bar{y}) - \bar{w}(0)]$, that is required for implementation of $(\ell,X)$. As one moves, say, from $X_o > X^*$ to $X_1 = X^*$ and from $X_1 = X^*$ to $X_2 > X^*$, first one raises and then one lowers $p(X)X$; at the same time one lowers $r(X)$. In terms of the original formulation (4*) or (13) of the contracting problem, one might suspect that a decrease of the term $r(X)$ in the entrepreneur's expected payoff might be disadvantageous. However, because the intercept of the incentive scheme, $\bar{w}(0)$, provides for side payments compensating the entrepreneur for adverse changes in $r(X)$, the entrepreneur's payoff expectation may be identified with the overall surplus $p(X)Xf(I,\ell) - \ell - I + A$ (see problem (19)), and the term $r(X)$ matters *only* as a higher or lower value of the expected reward for success makes it more or less difficult to meet the financiers' participation constraint: A decrease in $r(X)$ is advantageous because it reduces the cost of inducing the entrepreneur to choose the agreed pair of risk and effort levels. Any risk level $X < X^*$ is thus dominated by $X^*$, which involves a higher expected return as well as a lower implementation cost. For $X > X^*$, there is a tradeoff between the expected return $p(X)X$ and the term $r(X)$ in the implementation cost. The resolution of this tradeoff depends on the parameters of the problem, but, with $A < I^*$ and $p(X^*) + p'(X^*)X^* = 0$, it is never desirable to set $X = X^*$.

The second-best investment and effort levels are more difficult to characterize. Much depends on the way they interact in the production function. If one compares (20) and (21) with (10a) and (10b), one notices three differences:

- On the left-hand sides of (20) and (21), the expected marginal products of investment and effort involves the term $p(X)X$ rather than $p(X^*)X^*$. Since $p(X)X < p(X^*)X^*$, this would seem to indicate that $I$ and $\ell$ will be below the first-best levels.
- The choice of investment and effort levels takes account of the effects of $I$ and $\ell$ on the ratio $f(I,\ell)/f_2(I,\ell)$ in the incentive payment which the entrepreneur receives in the event of success. Thus the right-hand side of (20) may be positive or negative, and $p(X)X\ f_1$ may be greater or less than the marginal opportunity cost of funds, depending on whether an increase in $I$ raises or lowers the required incentive payment to the entrepreneur, making it harder or easier to meet the financiers' participation constraint. In (21), the term $1 - ff_{22}/f_2^2 > 1$ indicates

that the required incentive payment rises more than proportionately with $\ell$ because one must counter the decrease in the marginal product of effort.

- The term $1 - ff_{22}/f_2^2$ in (21) is premultiplied by r(X) which is less than one. *A priori* it is therefore not clear whether the marginal incentive cost of increasing $\ell$, is greater or less than one, the marginal social cost of increasing $\ell$. Depending on the difference, the right-hand side of (21) may be positive or negative, and $p(X)X f_2$ may be greater or less than the marginal social cost of increasing $\ell$.

Somewhat more can be said if the production function f is assumed to be *homothetic*, i.e., if f takes the form

$$f(I,\ell) = g(\psi(I,\ell)) \tag{24}$$

where g is increasing and strictly concave and $\psi$ is linearly homogeneous. In this case, (20) and (21) take the form

$$p(X)Xf_1(I,\ell) - 1 = \frac{\mu-1}{\mu} r(X) \frac{\psi_1(I,\ell)}{\psi_2(I,\ell)} \frac{g(\psi)}{g'(\psi)\psi} [\delta + \frac{\sigma-1}{\sigma}], \tag{25}$$

$$p(X)Xf_2(I,\ell) - 1 = \frac{\mu-1}{\mu}\left[\frac{W^*}{\ell} + r(X) \frac{g(\psi)}{g'(\psi)\psi} \left[\delta - \frac{I\psi_1(I,\ell)}{\ell\psi_2(I,\ell)} \frac{\sigma-1}{\sigma}\right]\right]. \tag{26}$$

In (25) and (26), $\delta$ is the elasticity of $g/g'\psi$ with respect to $\psi$, i.e.,

$$\delta := \frac{g'\psi}{g} - \frac{g''\psi}{g'} - 1 . \tag{27}$$

Further, $\sigma$ is the elasticity of substitution between I and $\ell$; since $\psi$ is linearly homogeneous, $\sigma = \psi_1\psi_2/\psi\psi_{12}$ (also $\sigma = - I\psi_1\psi_2/\psi\ell\psi_{22}$). Finally, $W^*$ is the net payoff expectation of the entrepreneur as indicated by (13) with $\bar{w}(0) = 0$; for f satisfying (24), this takes the form

$$W^* = \left[r(X) \frac{g}{g'\psi} \frac{\psi}{\psi_2\ell} - 1\right]\ell. \tag{28}$$

In conditions (25) and (26), the wedge between social marginal expected benefits and social marginal costs of increasing I or $\ell$ is given by the effects of these variables on the entrepreneur's payoff expectation $W^*$; this reflects the fact that $W^*$ is equal to the difference between the implementation cost and the social cost of the entrepreneur's effort. Three types of effects can be distinguished:

- As $W^*$ is positive, the term in square brackets in (28) is positive, so for given values of $g/g'\psi$ and $\psi/\psi_2\ell$ an increase in $\ell$ raises $W^*$ and makes it harder to meet the financiers' participation constraint. This effect is reflected in the first term on the right-hand side of (26). *Ceteris paribus* it makes for undereffort as one would expect.
- An increase in I or $\ell$ raises $\psi(I,\ell)$; this may affect the ratio $g/g'\psi$ in (28). If the elasticity $\delta$ in (25) and (26) is positive, the increase in $\psi$ raises $g/g'\psi$ and - *ceteris paribus* - $W^*$, making it harder to meet the financier's participation constraint. As g rises relative to $g'\psi$ (treating $\psi/\psi_2\ell$ as given), the temptation for the entrepreneur to substitute risk for effort becomes larger; to balance this effect the reward for success must be increased. *Ceteris paribus* then, both I and $\ell$ are the smaller the larger is $\delta$.
- A change in the ratio $I/\ell$ may change the ratio $\psi/\psi_2\ell$ in (28). The direction of this effect depends on whether the elasticity of substitution is greater or less than one. For $\sigma>1$, an increase in I or a decrease in $\ell$ raise $\psi/\psi_2\ell$ and - ceteris paribus - $W^*$, making it harder to meet the financiers' participation constraint; for $\sigma<1$, the effect is reversed. *Ceteris paribus,* this effect makes for *underinvestment and overeffort if* $\sigma>1$ and for *overinvestment and undereffort if* $\sigma<1$.

Depending on the signs of $\delta$ and $\sigma-1$, the different effects may or may not go in different directions. If they do go in different directions, there seems to be no reason why one effect should dominate another.[9] This suggests that no general results about second-best investment and effort levels are available.

Some progress can be made if one looks at the overall level of activity and the investment/effort ratio. Multiply (25) by I and (26) by $\ell$ and add the resulting equations. This yields

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9 Thus for the homogeneous CES specification $f(I,\ell) = (I^\alpha+\ell^\alpha)^{\beta/\alpha}$ with $\beta=9/10$ or larger, I have found that the right-hand side of (26) is always positive when $\alpha$ is close to zero or one ($\sigma$ close to one or very large), but for $\alpha=1/2$ ($\sigma=2$) and a suitably chosen success probability function, the right-hand side of (26) is negative.

(29) $$p(X)X[If_1(I,\ell) + \ell f_2(I,\ell)] - (I+\ell) = \frac{\mu-1}{\mu}\left[W^* + r(X)\,\frac{g}{g'\psi_2}\,\delta\right].$$

The left-hand side of (29) is the derivative of the function $\lambda \to p(X)Xf(\lambda I,\lambda\ell) - \lambda I - \lambda\ell$ at the point $\lambda=1$. Since $W^*$ is positive, the right-hand side is certainly positive if $\delta\geq 0$. For this case, (28) indicates that the second-best outcome involves underinvestment and undereffort in the sense that for given X and a given investment/effort ratio a small proportional increase in *both*, I and $\ell$, at the same time would raise expected surplus. From this observation one obtains:

PROPOSITION 5.4.
Assume A.1–A.4, let $\tilde{\theta}\equiv 1$, and suppose that f is homothetic. Let $(I,\ell,X,w(.),\bar{y},\bar{w}(.))$ be a solution to problem (4*). If $\delta\geq 0$, then $f(I,\ell) < f(\hat{I}(X),\hat{\ell}(X))$ where $(\hat{I}(X),\hat{\ell}(X)) := \arg\max_{I',\ell'} [p(X)Xf(I',l') - I' - \ell']$. If in addition $g'(\psi)\psi$ is increasing in $\psi$, then also $I+\ell < \hat{I}(X) + \hat{\ell}(X)$.

Since Proposition 5.3 implies $p(X)X < p(X^*)X^*$, one also obtains:

COROLLARY 5.5.
Under the assumptions of Proposition 5.4, any solution $(I,\ell,X,w(.),\bar{y},\bar{w}(.))$ to problem (4*) with $\delta\geq 0$ satisfies $f(I,\ell) < f(I^*,\ell^*)$. If in addition $g'(\psi)\psi$ is increasing in $\psi$, then also $I+\ell < I^*+\ell^*$.

Proposition 5.4 and its corollary cover, e.g., the case where f is *homogeneous*, i.e., where the function g(.) in (23) is given as $g(\psi) = \psi^{\beta}$ for some constant $\beta \in (0,1)$.[10] *For this case, the proposition and its corollary show that at a second-best contract the aggregate of investment and effort is inefficiently low.* It is not only lower than $I^* + \ell^*$, the first-best level, but also lower than $\hat{I}(X) + \hat{\ell}(X)$, the level that results from unconstrained surplus maximization when the (inefficient) risk choice X is taken as given.
In the more general case, when $\delta\geq 0$ and $g'\psi$ is not necessarily increasing in $\psi$, one still finds that *the second-best return $\bar{y} = Xf(I,\ell)$ in the event of success is lower than the return* $Xf(\hat{I}(X),\hat{\ell}(X))$ *that would result from surplus maximization with X taken as given.* (Since $X>X^*$, a comparison of $\bar{y}$ with the

[10] In the homogeneous case, $g'\psi\equiv\beta g$, hence $\delta\equiv 0$ and $g'\psi$ is increasing in $\psi$.

first-best return $\bar{y}^* = X^*f(I^*,\ell^*)$ is out of the question.)

Turning to the investment-effort ratio, I note that if one subtracts (25) from (26) and rearranges terms, one obtains

$$\text{(30)}\qquad \left[p(X)Xg' - \frac{\mu-1}{\mu}\, r(X)\, \frac{g}{g'\psi}\, \frac{\delta}{\psi_2}\right](\psi_2-\psi_1) = \frac{\mu-1}{\mu}\left[\frac{W^*}{\ell} - r(X)\, \frac{g}{g'\psi_2\ell}\, \frac{\sigma-1}{\sigma}\right].$$

From (29), the square bracket on the left-hand side of (30) is equal to $[I + \ell + (\mu-1)W^*/\mu]/\psi$, which is always positive, regardless of $\delta$. Therefore the sign of $\psi_2-\psi_1$ depends only on the square bracket on the right-hand side of (30), and one obtains:

PROPOSITION 5.6.
Assume A.1–A.4, let $\tilde{\theta}\equiv 1$, and suppose that f is homothetic. Let $(I,\ell,X,w(.),\bar{y},\bar{w}(.))$ be a solution to problem (4*). If $\sigma\leq 1$, then $I/\ell > \hat{I}(X)/\hat{\ell}(X)$. Alternatively, if $\sigma$ is sufficiently large, then $I/\ell < \hat{I}(X)/\hat{\ell}(X)$.

COROLLARY 5.7.
Under the assumptions of Proposition 5.6, any solution $(I,\ell,X,w(.),\bar{y},\bar{w}(.))$ with $\sigma\leq 1$ and $\delta\geq 0$ satisfies $\ell < \hat{\ell}(X)$ and hence $\ell < \ell^*$. Any solution with $\sigma$ sufficiently large and $\delta\geq 0$ satisfies $I < \hat{I}(X)$ and hence $I < I^*$.

For $\sigma\leq 1$ and $\delta\geq 0$, the three effects that I discussed above all serve to drive effort down, yielding an unambiguous *undereffort* result; however, the relation of I to $\hat{I}(X)$ (or $I^*$) is unclear. In contrast, for $\sigma$ sufficiently large and $\delta\geq 0$, one obtains an unambiguous *underinvestment* result, but the assessment of effort is unclear.

For additional insight about the role of substitutability between investment and effort, I consider the case where the elasticity of substitution is an exogenous constant. Given $\sigma \in \mathbb{R}_{++}$, I assume that f satisfies (24) with

$$\text{(31a)}\qquad \psi(I,\ell) = \bar{\psi}(I,\ell,(\sigma-1)/\sigma)$$

where $\bar{\psi}$ is a function on $\mathbb{R}^2_+ \times (-\infty,1)$ such that for some $a \in (0,1)$, $b=1-a$, and any $(I,\ell,\alpha)$,

$$(31b) \qquad \bar{\psi}(I,\ell,\alpha) = \begin{cases} [a^{1-\alpha}I^{\alpha} + b^{1-\alpha}\ell^{\alpha}]^{1/\alpha}, & \text{if } \alpha \in \mathbb{R}_{--} \cup (0,1), \\ \left(\frac{I}{a}\right)^{a}\left(\frac{\ell}{b}\right)^{b} & \text{if } \alpha = 0 \end{cases}$$

The parametrization here is chosen so that the first-best outcome is independent of $\alpha(\sigma)$ and $\sigma$. Specifically, one has $(I^*,\ell^*) = (aC^*,bC^*)$ where $C^*$ is the unique solution to the condition $p(X^*)X^* g'(C^*) = 1$. Moreover one easily verifies that $\bar{\psi}(I^*,\ell^*,\alpha) = C^*$ and $f(I^*,\ell^*) = g(C^*)$, regardless of $\alpha$ and $\sigma$.

PROPOSITION 5.8.

Assume A.1-A.4 and let $\tilde{\theta} \equiv 1$. For $\sigma \in \mathbb{R}_{++}$, let $(I(\sigma),\ell(\sigma),X(\sigma),w(.,\sigma),\bar{y}(\sigma),\bar{w}(.,\sigma))$ be a solution to problem (4*) when f satisfies (24), (31a), and (31b), and let $W^*(\sigma)$ be the corresponding payoff expectation of the entrepreneur. Then

$$\lim_{\sigma\to\infty} I(\sigma) = A, \quad \lim_{\sigma\to\infty} \ell(\sigma) = I^* + \ell^* - A, \quad \lim_{\sigma\to\infty} X(\sigma) = X^*,$$

$$\lim_{\sigma\to\infty} \bar{y}(\sigma) = \lim_{\sigma\to\infty} w(\bar{y}(\sigma),\sigma) = \lim_{\sigma\to\infty} \bar{w}(\bar{y}(\sigma),\sigma) = X^*f(I^*,\ell^*),$$

and

$$\lim_{\sigma\to\infty} W^*(\sigma) = p(X^*)X^*f(I^*,l^*) - I^* - \ell^*.$$

PROPOSITION 5.9.

Assume A.1-A.4 and let $\tilde{\theta} \equiv 1$. For $\sigma \in \mathbb{R}_{++}$, let $(I(\sigma),\ell(\sigma),X(\sigma),w(.,\sigma),\bar{y}(\sigma),\bar{w}(.,\sigma))$ be a solution to problem (4*) when f satisfies (24), (31a), and (31b), and let $W^*(\sigma)$ be the corresponding payoff expectation of the entrepreneur. Then

$$\lim_{\sigma\to 0} I(\sigma) = I^*, \quad \lim_{\sigma\to 0} \ell(\sigma) = \ell^*, \quad \lim_{\sigma\to 0} X(\sigma) = X^*, \quad \lim_{\sigma\to 0} \bar{y}(\sigma) = X^*f(I^*,\ell^*),$$

$$\lim_{\sigma\to 0} w(\bar{y}(\sigma),\sigma) = \lim_{\sigma\to 0} \bar{w}(\bar{y}(\sigma),\sigma) = X^*f(I^*,\ell^*) - \frac{I^*-A}{p(X^*)},$$

and

$$\lim_{\sigma\to 0} W^*(\sigma) = p(X^*)X^*f(I^*,\ell^*) - I^* - \ell^*.$$

Propositions 5.8 and 5.9 show that the agency problem is negligible and *payoff expectations are close to their first-best levels whenever investment and* *effort are close to being perfect substitutes or close to being perfect complements.*[11] To understand these results, it is useful to look at the limiting cases $\sigma=\infty$, $\alpha(\sigma) = 1$ and $\sigma=0$, $\alpha(\sigma) = -\infty$ even though these limiting cases are not covered by Assumption A.1.

When investment and effort are perfect substitutes, the agency problem disappears because the entrepreneur can always substitute effort for investment so no recourse to outside finance is needed. A first-best outcome is attained by setting I=A, $\ell=g'^{-1}(1/p(X^*)X^*) - A$, and $X=X^*$. As indicated by Proposition 5.8, this first best outcome is approximated when $\sigma$ is large and $\alpha(\sigma)$ is close to one.

When investment and effort are perfect complements, the matter is more complicated because a first-best outcome does require outside finance. However, in this case, $f(I,\ell) = g(\min(I/a,\ell/b))$, so for $I/a = \ell/b$, the partial derivative $f_2(I,\ell)$ is not defined: The right-hand derivative of f with respect to $\ell$ is zero, the left-hand derivative is $g'(\ell/b)/b$. Nevertheless one finds that Lemma 4.3 remains valid provided that for $I/a = \ell/b$, $f_2(I,\ell)$ in formula (11) is replaced by any number between 0 and $g'(\ell/b)/b$. To implement the first-best outcome $(I^*,\ell^*,X^*)$ it is thus necessary and sufficient to have

$$\bar{w}(X^*f(I^*,\ell^*)) - \bar{w}(0) \geq \frac{1}{(-p'(X^*)X^*)}\frac{f(I^*,\ell^*)}{g'(\ell^*/b)/b} = \frac{1}{p(X^*)}\frac{f(I^*,\ell^*)}{g'(\ell^*/b)} \text{ (1-a)}$$

$$= X^*f(I^*,\ell^*) - \frac{1}{p(X^*)}\frac{g(I^*/a)a}{g'(\ell^*/b)}.$$

Since $g(I^*/a)a/g'(I^*/a) > I^* \geq I^* - A$, it follows that the contract $(I^*,\ell^*,X^*,w(.),\bar{y},\bar{w}(.))$ with $\bar{y} = X^*f(I^*,\ell^*)$ and

$$w(y) = \bar{w}(y) = \begin{cases} X^*f(I^*,\ell^*) - (I^*-A)/p(X^*), & \text{if } y=\bar{y}, \\ 0, & \text{if } y\neq\bar{y}, \end{cases}$$

satisfies the constraints of problem (4*), so indeed the first-best outcome is attainable in the case of perfect complements. As indicated by Proposition 5.9, this first-best outcome is approximated for $\sigma$ close to zero.

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11 Proposition 5.8 also shows that the second-best effort level may indeed exceed $\ell^*$. Since $I^* > A$, for $\sigma$ sufficiently large, one has $\ell(\sigma) > \ell^*$ and $I(\sigma) < I^*$, so in comparison to the first-best outcome there is "excessive effort" compensating for the underinvestment. Whether for $\sigma$ close to zero one may also have $I(\sigma) > I^*$, I do not know.

Between the extremes of perfect substitutability and perfect complementarity, the relation between $\sigma$ and $W^*(\sigma)$ is quite complicated, involving multiple critical points for at least some specifications of g(.) and p(.). Even so, a simple revealed-preference argument shows that if for some $\sigma_o$ one has $I(\sigma_o)/\ell(\sigma_o) = a/b$, then $W^*$ attains a global minimum at $\sigma_o$. Since Proposition 5.6 implies $I(\sigma)/\ell(\sigma) > a/b$ for $\sigma \leq 1$ and $\lim_{\sigma \to \infty} I(\sigma)/\ell(\sigma) < a/b$, it follows that if $I(\sigma)$ and $\ell(\sigma)$ depend continuously on $\sigma$,[12] then $W^*$ attains a global minimum at some value of the elasticity of substitution $\sigma$ exceeding one where the investment/effort ratio is equal to the first-best ratio a/b. The point is that (31a) and (31b) imply $\psi(I,\ell) = \ell/b$ and $\psi_2(I,\ell) = 1$, if $I/\ell = a/b$ regardless of $\sigma$. Any contract $(I,\ell,X,w(.),\bar{y},\bar{w}(.))$ with $I/\ell = a/b$ which satisfies the constraints of problem (4*) for *some* value of $\sigma$ will therefore satisfy these constraints for *all* values of $\sigma$; moreover the entrepreneur's payoff expectation under such a contract is independent of $\sigma$. If such a contract is optimal at $\sigma_o$ but *not* at $\sigma_1 \neq \sigma_o$, it follows that $W^*(\sigma_1) > W^*(\sigma_o)$. Then at $\sigma_o$, $W^*$ must have a global minimum and the agency cost of outside finance must be maximal.

## 6. ANALYSIS OF THE MODEL: THE CASE OF NONDEGENERATE $\tilde{\theta}$

To complete the analysis, I consider the case where the disturbance term $\tilde{\theta}$ is a nondegenerate random variable. In this case, observation of the return $\tilde{y}$ in the event of success is not enough for the financiers to determine whether $\ell$ and X have been chosen so as to provide the desired return target $\bar{y}$ for the event of success. If the range of $\tilde{\theta}$ is equal to $\mathbb{R}_{++}$ then indeed no return realization at all permits any precise inference about $\bar{y}$.

Going back to the problem formulation (13), I note that the distribution of $\tilde{\theta}$ plays no role for the entrepreneur's objective (13), the financiers' participation constraint (14), or the entrepreneur's individual-rationality constraint (15). The distribution of $\tilde{\theta}$ does however play a role for the

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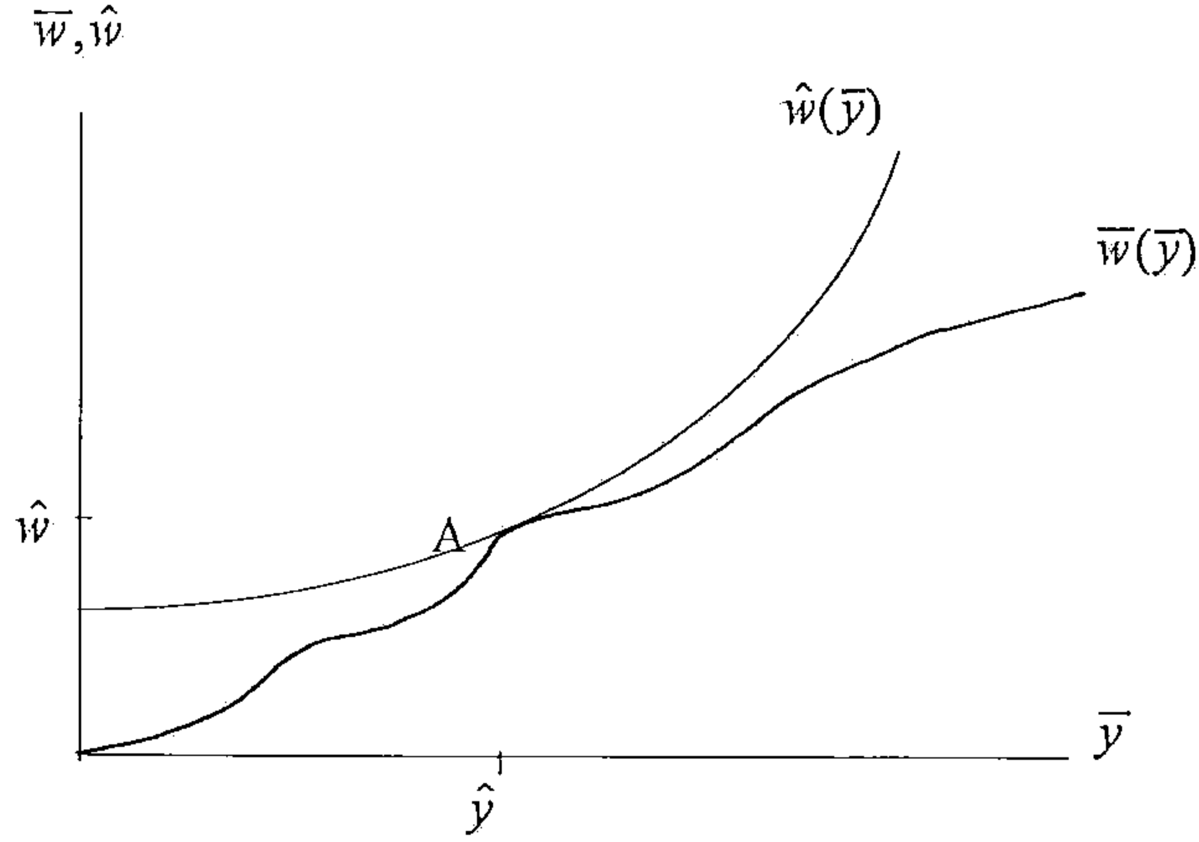
12 This in turn holds under suitable curvature and skewness conditions on g(.) and p(.), ensuring that for each $\sigma$ the solution to problem (4*) is unique.

FIGURE 1



$\overline{w}, \hat{w}$
$\hat{w}(\overline{y})$
$\overline{w}(\overline{y})$
$\hat{w}$
A
$\overline{y}$
$\hat{y}$

relation (8) between the incentive scheme $w(.)$ and the function $\bar{w}(.)$ which relates the entrepreneur's expected return in the event of success to the overall return target $\bar{y}$. Indirectly then the distribution of $\tilde{\theta}$ also matters for the incentive compatibility constraint (16) (respectively (6*b)).

In the case $\tilde{\theta} \equiv 1$, the latter constraints were trivially satisfied by setting $w(.) = \bar{w}(.)$ with $w(\bar{y}') = \bar{w}(\bar{y}') = 0$ for $\bar{y}' \neq \bar{y}$. In the nondegenerate case, this device is not available: In this case, (8) implies $w(.) \neq \bar{w}(.)$, and $\bar{w}(\bar{y}) > 0$ may imply $\bar{w}(\bar{y}') > 0$ for all $\bar{y}'$ in some neighborhood of $\bar{y}$. This raises the question whether the constraints (8) and (16) do have a material effect on second-best outcomes when $\tilde{\theta}$ is nondegenerate.[13]

The problem is illustrated in Figure 1. Let $(\hat{I},\hat{\ell},\hat{X})$ be any outcome that is to be implemented. This outcome determines a target $\hat{y} = \hat{X}f(\hat{I},\hat{\ell})$ for the conditional expectation of $\tilde{y}$ given the event of success. Through the incentive-compatibility condition (11) - with $\bar{w}(0) = 0$ -, it also determines a requisite value $\hat{w} = \bar{w}(\hat{y})$ for the conditional expectation of $\tilde{w}$ given the event of success. The pair $(\hat{y},\hat{w})$ is indicated by the point A in Figure 1. Assuming that $U^*(\hat{y},\hat{w},\hat{I}) > 0$, let $\hat{w}(.)$ indicate the entrepreneur's indifference curve through A, i.e., the set of all $(\bar{y},\bar{w})$ such that $U^*(\bar{y},\bar{w},\hat{I}) = U^*(\hat{y},\hat{w},\hat{I})$. Given the distribution F of the random variable $\tilde{\theta}$, implementation of $(\hat{I},\hat{\ell},\hat{X})$ requires that one find a function $w(.) \in \mathcal{W}$ such that

(32a) $$\bar{w}(\hat{y}) := \int w(\hat{y}\theta)dF(\theta) = \hat{w}$$

and, for all $\bar{y} \neq \hat{y}$,

(32b) $$\bar{w}(\bar{y}) := \int w(\bar{y}\theta)dF(\theta) \leq \hat{w}(\bar{y}).$$

As shown in Figure 1, (32a) and (32b) together require that the difference $\hat{w}(\bar{y}) - \bar{w}(\bar{y})$ attain a global minimum at $\bar{y}=\hat{y}$. The necessary first-order condition for this minimization is

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13 For a more general discussion of the problems caused by noise, see Caillaud, Guesnerie and Rey (1992) and the references given there. Because of the nonnegativity condition on w(.), the problem here is somewhat different though.

$$\text{(32c)} \qquad \frac{d\bar{w}}{d\bar{y}}(\hat{y}) = \frac{d\hat{w}}{d\bar{y}}(\hat{y})$$

if $\bar{w}(.)$ is differentiable at $\hat{y}$ (and the corresponding inequalities on right-hand and left-hand derivatives otherwise). The overall implementation problem may therefore be decomposed into two subproblems: First, does there exist a function $w(.) \in \mathcal{W}$ that satisfies (32a) and (32c)? Second, if a function $w(.) \in \mathcal{W}$ satisfies (32a) and (32c), does it also satisfy (32b)?

I begin with a negative result showing that for some distributions of $\tilde{\theta}$, an outcome $(\hat{I},\hat{\ell},\hat{X})$ cannot be implemented if $\hat{X}$ is too large.

PROPOSITION 6.1.

Assume A.1-A.4, and suppose that the distribution of $\tilde{\theta}$ takes the form

$$\text{(33)} \qquad F(\theta) = \begin{cases} \frac{k}{k+1}\,(\theta/\bar{\theta}) & \text{, if } \theta \le \bar{\theta} \\ 1 - \frac{1}{k+1}(\theta/\bar{\theta})^{-k} & \text{, if } \theta > \bar{\theta} \end{cases}$$

where $k > 1$ and $\bar{\theta} := (k-1)/2k$. Let $(\hat{I},\hat{\ell},\hat{X})$ be any outcome satisfying $1/r(\hat{X}) > k$, let $\hat{y} = \hat{X}f(\hat{I},\hat{\ell})$, $\hat{w} = f(\hat{I},\hat{\ell})/(-p'(\hat{X})\hat{X}f_2(\hat{I},\hat{\ell})$ and suppose that $U^*(\hat{y},\hat{w},\hat{I}) > 0$. For any $w(.) \in \mathcal{W}$, the function $\bar{w}(.)$ that is defined by $w(.)$ through (8) cannot satisfy *both*, (32a) *and* (32c). No contract $(\hat{I},\hat{\ell},\hat{X},w(.),\hat{y},\bar{w}(.))$ with $1/r(\hat{X}) > k$ can satisfy all the constraints of problem (4*).

The logic of this result is simple. If F takes the form (33), then for *any* $w(.) \in \mathcal{W}$, the function $\bar{w}(.)$ that is defined by (8) has an elasticity that does not exceed k; one has

$$\text{(34a)} \qquad \frac{d\bar{w}}{d\bar{y}} \le k\,\frac{\bar{w}(\bar{y})}{\bar{y}} ,$$

for all $\bar{y}$. On the other hand, the indifference curve $\hat{w}(.)$ satisfies

$$\text{(34b)} \qquad \frac{d\hat{w}}{d\bar{y}}(\hat{y}) = \frac{1}{r(\hat{X})}\,\frac{\hat{w}}{\hat{y}} .$$

Upon comparing (34a) and (34b), one sees that if $1/r(\hat{X}) > k$, then either

$\frac{d\bar{w}}{d\bar{y}}(\hat{y}) < \frac{d\hat{w}}{d\bar{y}}(\hat{y})$ or $\bar{w}(\hat{y}) > \hat{w}(\hat{y})$ (or both), so either (32c) or (32a) (or both) must be violated. Therefore, no incentive scheme $w(.) \in \mathcal{W}$ will serve to implement an outcome that involves such a risk choice $\hat{X}$.

Given Proposition 5.3, Proposition 6.1 immediately yields:

COROLLARY 6.2.
Assume A.1–A.4, and let $(I,\ell,X,\mu)$ be a solution to problem (19). If the distribution of $\tilde{\theta}$ takes the form (33) with k sufficiently close to one, then there exist no functions $w(.) \in \mathcal{W}$ and $\bar{w}(.)$ such that $(I,\ell,X,w(.),Xf(I,\ell),\bar{w}(.))$ satisfies the constraints of problem (4*).

Corollary 6.2 shows that even though all parties are risk neutral, certain forms of nondegeneracy of $\tilde{\theta}$ may have substantive effects on second-best outcomes. The reason is that if the distribution of $\tilde{\theta}$ is very flat (close to "the uniform distribution on $\mathbb{R}_+$"), then regardless of what incentive scheme $w(.) \in \mathcal{W}$ is used the elasticity of the conditional expectation of $\tilde{w}$ given the event of success with respect to $\bar{y}$ is close to one. This means that one cannot lower the entrepreneur's share $\bar{w}(\bar{y})/\bar{y}$ without at the same time reducing the slope $\frac{d\bar{w}}{d\bar{y}}$ and thereby weakening incentives for choosing risk and effort levels to attain the given target for $\bar{y}$. The link between the slope $\frac{d\bar{w}}{d\bar{y}}$ and the level $\bar{w}/\bar{y}$ that is provided by (33a) exacerbates the conflict between the need to provide a proper share of returns to the financiers and the need to provide proper incentives to the entrepreneur.

Distributions of the form (33) involve hazard rates going to zero as $\theta$ goes out of bounds. They are thus quite special. Under more conventional assumptions about hazard rates, the problem discussed in Proposition 6.1 cannot arise. Indeed one has:

LEMMA 6.3.
Assume A.1–A.4. For $\rho>0$, let

$$\hat{\theta}(\rho) := \begin{cases} \int_{\rho-}^{\infty} \theta dF(\theta)/(1-F(\rho-)) , & \text{if } F(\rho-) < 1 , \\ \rho , & \text{if } F(\rho) = 1 , \end{cases}$$

and suppose that $\lim_{\rho\to\infty} \hat{\theta}(\rho)/\rho = 1$. For any outcome $(\hat{I},\hat{\ell},\hat{X})$ with $U^*(\hat{y},\hat{w},\hat{I}) > 0$, where $\hat{y} = \hat{X}f(\hat{I},\hat{\ell})$, $\hat{w} = f(\hat{I},\hat{\ell})/(-p'(\hat{X})\hat{X}f_2(\hat{I},\hat{\ell}))$, there exist constants $\alpha$ and R such that for $w(.) \in \mathcal{W}$ satisfying

(35) $$w(y) = (1-\alpha) \max (y-R,0)$$

for all $y \geq 0$, the function $\bar{w}(.)$ that is defined by $w(.)$ through (8) satisfies (32a) and (32c). If $r(\hat{X}) < 1$, the constants $\alpha$ and R can be chosen so that for all $\bar{y} > 0$,

(36) $$\ln \bar{w}(\bar{y}) \leq \ln \hat{w} + [\ln \bar{y} - \ln \hat{y}]/r(\hat{X}).$$

REMARK 6.4.
If F has a density $\varphi$, the condition $\lim_{\rho\to\infty} \hat{\theta}(\rho)/\rho = 1$ is implied by the condition $\lim_{\theta\to\infty} \theta\varphi(\theta)/(1-F(\theta)) = \infty$.

For the case $r(\hat{X}) < 1$, the logic behind Lemma 6.3 is illustrated in Figure 2. Note that for $w(.) \in \mathcal{W}$ satisfying (35) with R>0, one has

(37a) $$\bar{w}(\bar{y}) = (1-\alpha)\bar{y} \int_{\rho}^{\infty} \theta \, dF(\theta) - (1-\alpha)R(1-F(\rho))$$

and

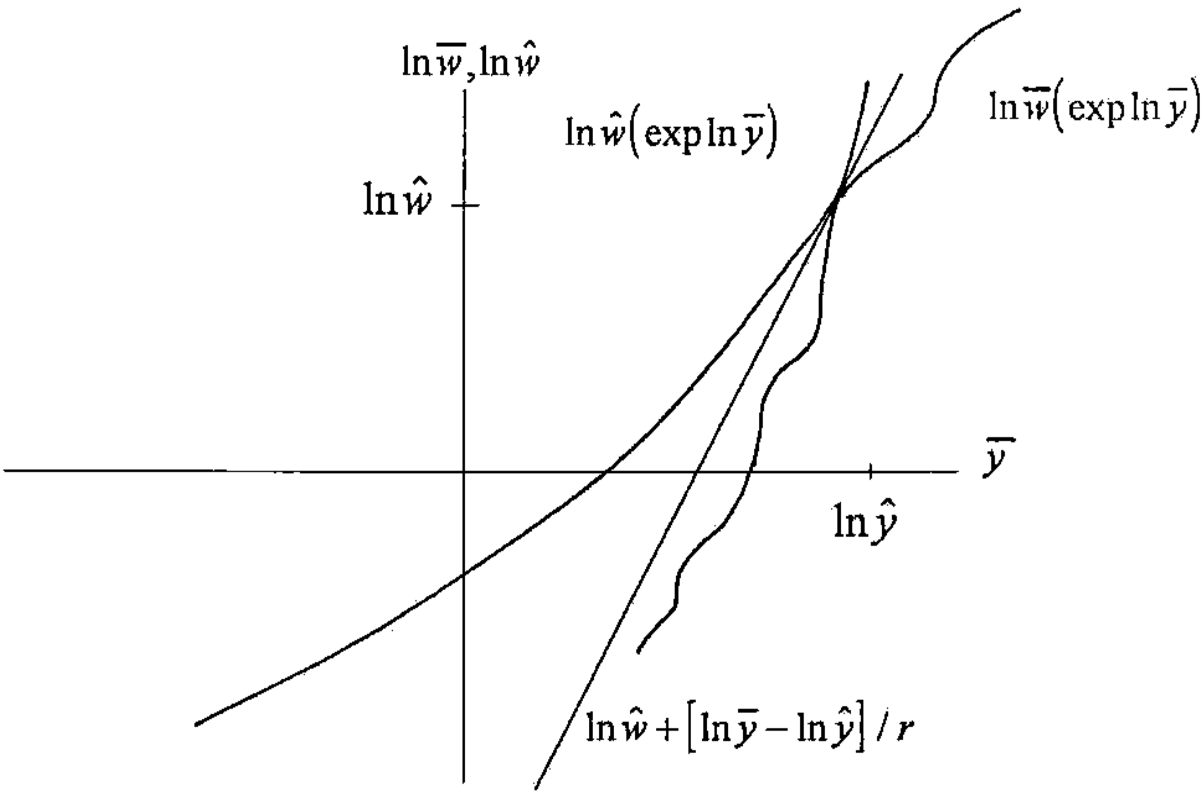
(37b) $$\frac{d\bar{w}}{d\bar{y}} (\bar{y}-) = (1-\alpha) \int_{\rho-}^{\infty} \theta \, dF(\theta)$$

where $\rho = R/\bar{y}$. The (left-hand) elasticity of $\bar{w}(.)$ with respect to $\bar{y}$ is thus

(37c) $$\frac{\bar{y}}{\bar{w}(\bar{y})} \frac{d\bar{w}}{d\bar{y}} (\bar{y}-) = \frac{\hat{\theta}(R/\bar{y})}{\hat{\theta}(R/\bar{y})-R/\bar{y}} .$$

If one looks at the problem in terms of $\ln \bar{w}$ and $\ln \bar{y}$, rather than $\bar{w}$ and $\bar{y}$, one easily sees that for $s = \ln \bar{y}$, the elasticity (37c) corresponds to the slope of the function $s \to \ln \bar{w}(e^s)$, i.e. one has

FIGURE 2


$\ln\overline{w}, \ln\hat{w}$
$\ln\hat{w}\left(\exp\ln\overline{y}\right)$
$\ln\overline{w}\left(\exp\ln\overline{y}\right)$
$\ln\hat{w}$
$\overline{y}$
$\ln\hat{y}$
$\ln\hat{w} + \left[\ln\overline{y} - \ln\hat{y}\right] / r$

$$(37^{*}c) \qquad \frac{d}{ds} \ln \bar{w}(e^{s}) = \frac{\hat{\theta}(\exp(\ln R - s))}{\hat{\theta}(\exp(\ln R - s)) - \exp(\ln R - s)}.$$

Given the function $\hat{\theta}(.)$, (37*c) defines a two-parameter family of functions, the two degrees of freedom corresponding to the choice of ln R and the choice of the constant of integration (reflecting the underlying choice of $\alpha$). In terms of a $\ln\bar{y} - \ln\bar{w}$ - diagram, the two degrees of freedom (and hence the underlying choices of R and $\alpha$) have a simple geometric interpretation: changes in ln R correspond to horizontal shifts in the solution curve to (37*c), changes in the constant of integration as usual correspond to vertical shifts in the solution curve to (37*c).

In terms of this geometric picture, the conclusion of Lemma 6.3 amounts to the claim that the two degrees of freedom can be used so that (i) the solution curve to (37*c) lies nowhere above the straight line $\ln \hat{w} + [\ln \bar{y} - \ln \hat{y}]/r(\hat{X})$ with slope $1/r(\hat{X})$ through the point $(\ln \hat{y}, \ln \hat{w})$ and (ii) this curve touches the straight line $\ln \hat{w} + [\ln \bar{y} - \ln \hat{y}]/r(\hat{X})$ in the point $(\ln \hat{y}, \ln \hat{w})$, as shown in Figure 2. For this claim to be valid, it is necessary and sufficient that the solutions to (37*c) have a slope less than $1/r(\hat{X})$ for s sufficiently large and a slope greater than $1/r(\hat{X})$ for s sufficiently small. The former requirement holds automatically because $\hat{\theta}(0) = E\tilde{\theta} = 1$; the latter requirement is assured by the assumption that $\lim_{\rho\to\infty} \hat{\theta}(\rho)/\rho = 1$, which yields $\lim_{s\to\infty} \hat{\theta}(\exp(\ln R-s))/(\hat{\theta}(\exp(\ln R-s)) - \exp(\ln R-s)) = \infty$.

The main result of this section is now stated as:

PROPOSITION 6.5.
Assume A.1-A.4, and suppose that $\lim_{\rho\to\infty} \hat{\theta}(\rho)/\rho = 1$. Suppose further that f is homothetic, that the elasticity $\delta$ defined in (27) is everywhere nonnegative, and that the elasticity of substitution $\sigma$ between investment and effort is nowhere greater than one. If $(I,\ell,X,\mu)$ is a solution to problem (19), there exist constants $\alpha < 1$ and $R > 0$ such that for w(.) given by (35) and $\bar{w}(.)$ given by (8), the contract $(I,\ell,X,w(.), Xf(I,\ell),\bar{w}(.))$ is a solution to problem (4*).

In contrast to Corollary 6.2, Proposition 6.4 exhibits a *class of cases in* *which the nondegeneracy of* $\tilde{\theta}$ *has no substantive effects on second-best outcomes.* Two sets of conditions are imposed: (i) As discussed in Proposition 6.1 and Lemma 6.3, the condition $\lim_{\rho\to\infty} \hat{\theta}(\rho)/\rho = 1$ ensures that the tangency conditions (32a) and (32c) have a solution, which moreover satisfies (36). (ii) The additional conditions on f control the curvature of the indifference curve $\hat{w}(.)$. They imply that along the indifference curve $\hat{w}(.)$, the risk level that results from the maximization in (9) is increasing in $\bar{y}$. In view of (34b), this in turn implies that in the relevant range, with r(X)<1, the elasticity of $\hat{w}$ is increasing in $\bar{y}$, so ln $\hat{w}$ may be written as a *convex* function of ln $\bar{y}$.

As illustrated in Figure 2, convexity of ln $\hat{w}$ in ln $\bar{y}$ implies that

$$\ln \hat{w}(\bar{y}) \geq \ln \hat{w}(\hat{y}) + [\ln \bar{y} - \ln \hat{y}]/r(\hat{X}),$$

i.e., in a $\ln\bar{y}$ - $\ln\hat{w}$ - diagram, the indifference curve lies nowhere below the straight line with slope $1/r(\hat{X})$ through (ln $\hat{y}$, ln $\hat{w}$), and it touches this straight line in (ln $\hat{y}$, ln $\hat{w}$). Given that $\bar{w}(.)$ satisfies (36), it follows that ln $\hat{w}(\bar{y}) \geq$ ln $\bar{w}(\hat{y})$ and hence $\hat{w}(\bar{y}) \geq \bar{w}(\bar{y})$ for all $\bar{y}$, as required in (32b). The additional assumptions on f thus ensure that if $\alpha$ and R are chosen so that w(.) and $\bar{w}(.)$ satisfy (32a), (32c), and (36) for the desired outcome $(I,\ell,X)$, then the global minimization condition (32b) is also satisfied, and the incentive scheme w(.) does indeed serve to implement the outcome $(I,\ell,X)$. It is of interest to note that these assumptions on f are exactly the assumptions on which in Corollary 5.7 were shown to imply $\ell < \hat{\ell}(X)$ and $\ell<\ell^*$.

In what sense can the incentive scheme w(.) in Proposition 6.4 be interpreted as a result of the entrepreneur's issuing some mix of standard financial instruments? The parameter R resembles a standard debt obligation. The entrepreneur's receiving nothing for $\tilde{y} < R$ is reminiscent of the usual stipulations for bankruptcy. With this interpretation, it is of interest to note that R must be strictly positive, i.e., *an optimal second-best contract always involves an element of debt finance. This result is actually a corollary of the result that a second-best outcome always involves excessive risk taking.* To see this, note that for the solution curve to (37*c) to be tangent to the straight line ln $\hat{w}$ + [ln $\bar{y}$ - ln $\hat{y}$]/r($\hat{X}$) in the point (ln $\hat{y}$, ln $\hat{w}$) in Figure 2, one must have

$$(38) \qquad \frac{\hat{\theta}(R/\hat{y})}{\hat{\theta}(R/\hat{y}) - R/\hat{y}} = \frac{1}{r(\hat{X})},$$

so the positivity of R goes with the inequality $r(\hat{X}) < 1$. There is a direct link between the desirability of excessive risk taking for a second-best contract and the observation of Jensen and Meckling (1976) that debt finance may be desirable because it reduces the entrepreneur's share $\bar{w}/\bar{y}$ of (conditionally) expected returns without reducing the slope $\frac{d\bar{w}}{d\bar{y}}$ and thereby weakening incentives for effort-taking.[14]

The "share" parameter $\alpha$ is more difficult to interpret. From (37b), (32c), and (34b), one computes

$$(39) \qquad \alpha = 1 - 1 \Big/ p(X)Xf_2(I,\ell) \int_{\rho}^{\infty} \theta \, dF(\theta)$$

As indicated by equation (26), the assumptions of Proposition 6.5 imply $p(X)Xf_2(I,\ell) > 1$, so (39) implies $\alpha > 0$ if $\int_{\rho}^{\infty} \theta \, dF(\theta)$ is close to one, *but* $\alpha < 0$ if $\int_{\rho}^{\infty} \theta \, dF(\theta)$ is close to zero. Formally, one has

REMARK 6.6.

If the variance of $\tilde{\theta}$ is sufficiently small, the constant $\alpha$ in Proposition 6.5 is positive, and the incentive scheme w(.) may be interpreted as the result of the entrepreneur's issuing debt and equity in a mix involving the debt obligation R and the outside equity share $\alpha$.

REMARK 6.7.

If for some $\varepsilon > 0$, $F(1-r(X)) - F(\varepsilon)$ is sufficiently close to one, the constant $\alpha$ in Proposition 6.5 is *negative*, and the incentive scheme w(.) may *not* be interpreted as the result of the entrepreneur's issuing a suitable mix of debt and equity.

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[14] These considerations extend to the case of Proposition 6.1 and Corollary 6.2. For a given distribution F taking the form (33), for f homothetic, with $\delta \geq 0$ and $\sigma \leq 1$, and for $(I,\ell,X,\mu)$ solving problem (19) with the additional constraint $r(X) \geq 1/k$, the conclusion of Proposition 6.5 is again valid, and of course one has $X > X^*$, $r(X) < 1$, and $R > 0$.

The incentive effects of having the entrepreneur retain a share $1-\alpha$ of "equity" are blunted because the marginal effects of an increase in $\bar{y}$ do not entirely accrue to "equity holder". The "bankruptcy portion" $\int_0^{\rho} \theta \, dF(\theta)$ of the total effect $\int_0^{\infty} \theta \, dF(\theta) = 1$ of a unit increase in $\bar{y}$ accrues to "creditors". If a given target $\bar{y}$ is to be implemented anyway, the entrepreneur's "equity share" $1-\alpha$ must be adjusted upward to compensate for this effect. The amount of adjustment that is needed depends on the distribution of $\tilde{\theta}$. If the variance of $\tilde{\theta}$ is small, $\int_{\rho}^{\infty} \theta \, dF(\theta)$ is close to one, so not much of an adjustment is needed, and with $p(X)Xf_2(I,\ell) > 1$, an interpretation of the second-best contract in Proposition 6.4 in terms of standard debt and equity instruments is alright.

In contrast, if $\int_{\rho}^{\infty} \theta \, dF(\theta)$ is close to zero, the entrepreneur's "equity share" $1-\alpha$ under the second-best contract in Proposition 6.5 must be very large. In particular, $1-\alpha$ must eventually exceed 100%, at which point the financiers would have to hold negative "equity". Alternatively, if one thinks of the financiers holding all the equity and the entrepreneur holding a call option to buy a share $1-\alpha$ of the equity at the exercise price R per unit, then for $\int_{\rho}^{\infty} \theta \, dF(\theta)$ close to zero, the second-best contract in Proposition 6.5 invovles an equity obligation of the financiers under the call option, which is strictly greater than the equity position they hold. Under either arrangement, from a traditional finance perspective the second-best contract in proposition 6.4 seems rather outlandish.

One may object that for $\int_{\rho}^{\infty} \theta \, dF(\theta)$ close to zero, the second-best contract in Proposition 6.5 cannot be taken seriously as the analysis has not taken account of possible limitations on the financiers' ability or willingness to provide the entrepreneur with a return $\tilde{w}$ above the firm's return $\tilde{y}$. This objection is valid and important, but somewhat besides the point under discussion here. If one imposes, e.g., the additional constraint that $w(y)$ must not exceed $y$ for any $y$, this constraint rules out an incentive scheme of the piecewise linear form (35) with $\alpha<0$, but it does *not* eliminate the desire to raise the sensitivity of $\tilde{w} = w(\tilde{y})$ with respect to $\tilde{y}$ in order to compensate for the blunting of incentives that is due to the noise. The related analysis of Innes (1990) or Dionne and Viala (1992) suggests that such considerations should lead to contracts involving *discontinuous* incentive schemes of the form $w(y) = 0$ for $y<y^*$, $w(y) = y$ for $y \geq y^*$, the jump at $y^*$ making an extra contribution to the slope of the conditional-expectation function $\bar{w}(.)$. In other words, the addition of the constraint $w(y) \leq y$ for all y is likely to lead us further away from standard finance contracts, into the world of abstract incentive schemes.[15]

Doubts about the relation between incentive contracting and financial packaging may go even further. Even the rather comforting conclusion of Remark 6.6 seems to be due to a certain arbitrariness of optimal incentive schemes under risk neutrality rather than any substantive virtues of piecewise linear schemes, let alone debt and equity instruments. Using the same arguments as in the proofs of Lemma 6.3 and Proposition 6.5, one can in fact show that these results remain valid if the piecewise linear form (35) is replaced by the piecewise quadratic form

(35*) $$w(y) = (1-\alpha) \max [y-R-\beta(y-R)^2,0],$$

where $\beta$ is an arbitrary but fixed positive number.

I also suspect that piecewise linear incentive schemes of the form (35) may actually *not* be optimal in certain cases where the additional conditions on f that are imposed in Proposition 6.4 are not satisfied and the entrepreneur's indifference curve is *not* given by a convex function in $\ln \bar{y}$ - $\ln \bar{w}$ - space. I conjecture that as long as $\lim_{\rho\to\infty} \hat{\theta}(\rho)/\rho = 1$, the second-best outcomes of the model without noise can still be implemented, but this may require "more curvature" in the incentive scheme than is available with piecewise linearity, e.g., a scheme of the form (35*) with $\beta$ sufficiently large. At this point though I do not have any definite results without additional assumptions about f or about the shape of the indifference curve $\hat{w}$.

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15 Innes (1990) proposes to counter this effect by adding the stronger requirement that $y-w(y)$ be nondecreasing in y. I suspect that in the case of Remark 6.7 as in Innes's own analysis, this constraint would lead to a standard debt contract, i.e., a scheme of the form (35) with $\alpha=0$, being second-best, though not uniquely so. The problem is that we have no strong arguments for requiring monotonicity of $y-w(y)$ in y.

# 7. CONCLUDING REMARKS

In going through the analysis of this paper, I was most surprised by the difference between the results on second-best risk levels and the results on second-best effort and investment levels. On the one hand a clearcut general statement that second-best contracts involve excessive risk taking, on the other hand a complicated analysis with lots of ifs and buts before one can say anything about second-best investment and effort levels - the contrast is rather striking. One may however wonder how significant this finding is. One possibility would be that the difference between the results on second-best risk levels and second-best effort levels is merely an artefact of the special functional form that I have used. After all, there is already an asymmetry in the way in which $X$ and $\ell$ enter the expression $Xf(I,\ell)$. Nevertheless, I believe that the functional form has nothing to do with the matter. Instead the *asymmetry of results about second-best risk, effort, and investment levels seems to reflect a deeper asymmetry between these variables when seen from the perspective of the financiers.*

From the perspective of the financiers, the critical variables are $\pi=p(X)$, the success probability, $\bar{y} = Xf(I,\ell)$, the conditional expectation of returns given the event of success, and I, the investment level. They care neither about X nor about $\ell$ *except* as X and $\ell$ affect $\pi$ and $\bar{y}$. Hence from the financiers' point of view, a contract determines a triple $(\pi,\bar{y},I)$; the fact that this requires the entrepreneur to choose[16] $X = p^{-1}(\pi)$ and $\ell = \lambda(\pi,\bar{y},I)$ doesn't concern the financiers except for incentive compatibility considerations. Incentive compatibility considerations however will differ for $\pi$, $\bar{y}$, and I as these variables are not equally observable:

- The success probability $\pi$ is not observable at all; nor is there any observable variable which permits an inference about $\pi$.
- The return target $\bar{y}$ for the event of success is correlated with the actual return $\tilde{y}$ in the event of success; indeed in the absence of noise, $\bar{y}$ can be fully inferred from $\tilde{y}$ in the event of success.
- The investment level I is directly observable.

I believe that the differences in results about second-best risk, effort, and

16 Here, $\lambda(.)$ is the implicit function given by setting $\bar{y} \equiv p^{-1}(\pi)f(I,\lambda(\pi,\bar{y},I))$.

investment levels mirror these differences in observability of the variables $\pi$, $\bar{y}$, I, which the financiers care about.

To see the argument, take another look at the case $\tilde{\theta} \equiv 1$ that was studied in Section 5. In terms of the variables $\pi$, $\bar{y}$, I, the incentive compatibility condition (11) takes the form:

$$\bar{w}(\bar{y}) - \bar{w}(0) = \lambda_\pi(\pi,\bar{y},I),$$

so problems (19) becomes

$$\text{(40)} \qquad \underset{\pi,\bar{y},I}{\text{Max}} \ \underset{\mu\geq 1}{\text{Min}} \left[\pi\bar{y} - I - \lambda(\pi,\bar{y},I) + A + (\mu-1)(\pi\bar{y} - I - \pi\lambda_\pi(\pi,\bar{y},I) + A)\right],$$

with first-order conditions

$$\text{(41)} \qquad \bar{y} - \lambda_\pi = \frac{\mu-1}{\mu}\,\pi\lambda_{\pi\pi},$$

$$\text{(42)} \qquad \pi - \lambda_{\bar{y}} = \frac{\mu-1}{\mu}\,(\pi\lambda_{\pi\bar{y}} - \lambda_{\bar{y}})$$

$$\text{(43)} \qquad -1 - \lambda_I = \frac{\mu-1}{\mu}\,(\pi\lambda_{\pi I} - \lambda_I).$$

One easily verifies that $\lambda_{\pi\pi} > 0$, so (41) implies $\bar{y} - \lambda_\pi > 0$. Even without exploiting the properties of $\lambda$ that stem from the separable specification $\bar{y} = Xf(I,\ell)$, one thus finds that at the second-best $(\pi,\bar{y},I)$, risk taking is excessive in the local sense that a small increase in the success probability $\pi$ would raise the surplus expectation $\pi\bar{y} - I - \lambda$. Presumably this conclusion extends to all specifications for which the effort cost function $\lambda$ is increasing and convex in $\pi$ in the relevant range so that a first-order approach to the incentive problem can be used.

In contrast to the condition for $\pi$, the conditions for $\bar{y}$ and I, (42) and (43), do not involve any incentive considerations concerning these variables themselves. Instead these conditions reflect the consideration that $\bar{y}$ and I may affect the cost $\pi\lambda_\pi$ of providing proper incentives for the choice of $\pi$. Their formal structure is therefore not unlike the formal structure of Ramsey-Boiteux conditions for optimal commodity taxes. Perhaps the impenetrability of conditions (20) and (21) is merely an analogue of the difficulties that one encounters if one tries to rephrase Ramsey-Boiteux elasticity formulae in terms of the underlying preferences and technologies.

Form the perspective of this discussion, *the excessive-risk-taking result (Proposition 5.3) is in fact an undereffort result:* Given the levels of $\bar{y}$ and I, effort is set a level where a small increase would raise the success probability enough to make the overall expected surplus go up. The traditional distinction between moral hazard with respect to effort and moral hazard with respect to risk taking, which is natural from the perspective of the entrepreneur, is thus replaced by the distinction between moral hazard with respect to the success probability $\pi$ and moral hazard with respect to the expected return $\bar{y}$ in the event of success, the effort variable taking the back seat as it is driven by the return variables $\pi$ and $\bar{y}$.

In this formulation, one has the usual monotonicity relation between effort and either of the two return variables $\pi$ and $\bar{y}$. However, as the space of attainable return patterns is two-dimensional, there is no first-order dominance relation between *all* the return patterns associated with one effort level and *all* the return patterns associated with another effort level. In the analysis here, the dimension of the moral hazard problem was essentially reduced to the success probability dimension through the assumptions of noiseless observability of $\bar{y}$ and risk neutrality of the entrepreneur. The question is what happens in the more general case where the noise variable $\tilde{\theta}$ is nondegenerate, the entrepreneur is risk averse, and the problem studied here is compounded by risk sharing considerations. I conjecture that in this more general case, the differences in observability between $\pi$ and $\bar{y}$, the success probability and the expectation of returns in the event of success, will again play a role, implying that (i) the second-best level of $\pi$ is unambiguously too low, at least in the local sense that *ceteris paribus* an increase in $\pi$ would raise surplus, and (ii) the determination of the second-best level of $\bar{y}$ is encumbered by the effects of $\bar{y}$ on the incentive costs of implementing the desired success probability $\pi$.

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## APPENDIX: P R O O F S

PROOF of PROPOSITION 4.1

I need to give separate arguments for the case $A > 0$ and the case $A = 0$.

*CASE 1: A>0*

In the absence of outside finance, i.e., with $I \le A$, the entrepreneur can do no better than to set $I=A$, $\ell=\hat{\ell} := \text{argmax } p(X^*)X^*f(A,\ell) - \ell$, $X=X^*$, $\bar{y}=X^*f(A,\hat{\ell}(A))$, and $w(y) = \bar{w}(y) = y$ for all y. Hence it suffices to show that this possibility is strictly dominated by some contract $(I,\ell,X,w(.),\bar{y},\bar{w}(.))$ with $I > A$, which satisfies (5*), (6*), (7) and (8).

For $I>0$, and $\ell>0$, define

$$\text{(A.1)} \qquad \chi(I,\ell) := \left[1 - \frac{I-A}{p(X^*)X^*f(I,\ell)}\right]p(X^*)X^*f_2(I,\ell) - 1,$$

and note that, by the first-order condition for $\hat{\ell}$, $\chi(A,\hat{\ell}) = 0$. Moreover on some open neighborhood of the point $(A,\hat{\ell})$, the function $\chi$ is continuously differentiable, with $D_\ell\chi(A,\hat{\ell}) = p(X^*)X^*f_{22}(A,\hat{\ell}) < 0$. By the implicit function theorem, it follows that there exists a continuously differentiable real-valued function $\lambda$, defined on some neighborhood $(A-\varepsilon,A+\varepsilon)$ of A, such $\lambda(A) = \hat{\ell}$ and moreover $\chi(I,\lambda(I)) = 0$ for all $I \in (A-\varepsilon,A+\varepsilon)$. One easily checks that for $I \in (A-\varepsilon,A+\varepsilon)$, the contract $(I,\lambda(I),X^*,w(.),X^*f(I,\lambda(I)),\bar{w}(.))$ with $w(y) = \bar{w}(y) = [1-(I-A)/p(X^*)X^*f(I,\lambda(I))]y$ for all y satisfies the constraints (5*), (6*), (7), and (8). The entrepreneur's payoff under this contract is

$$\text{(A.2)} \qquad \zeta(I) := p(X^*)X^*f(I,\lambda(I)) - (I-A) - \lambda(I).$$

Since $\lambda(.)$ is continuously differentiable on $(A-\varepsilon,A+\varepsilon)$, $\zeta(.)$ is also continuously differentiable on $(A-\varepsilon,A+\varepsilon)$ and

$$\zeta'(A) = p(X^*)X^*f_1(A,\hat{\ell}) - 1 > 0,$$

so for I exceeding A, but sufficiently close to A, the contract $(I,\lambda(I),X^*,$ $w(.),X^*f(I,\lambda(I)),\bar{w}(.))$ with $w(y) = \bar{w}(y) = [1-(I-A)/p(X^*)X^*f(I,\lambda(I))]y$ for all y dominates the best possible outcome without outside finance.

*Case 2: A=0*

If A=0, I again claim that for I>0, sufficiently close to zero, there exists $\lambda(I)$ such that in (A.1), with A=0, one has $\chi(I,\lambda(I)) = 0$. To establish this claim, consider the investment and effort levels $I_o$ and $\ell_o$ specified in Assumption A.1. For I > 0, define $\delta(I) = I/I_o$, and note that Assumption A.1 implies

$$\chi(I,\delta(I)\ell_o) = \left[1 - \frac{I}{p(X^*)X^*f(I,\delta(I)\ell_o)}\right]p(X^*)X^*f_2(I,\delta(I)\ell_o) - 1$$

$$\geq p(X^*)X^*f_2(\delta(I)I_o,\delta(I)\ell_o) - \frac{I}{\delta(I)\ell_o} - 1$$

$$> 0$$

if I and $\delta(I)$ are sufficiently close to zero. For $\bar{\delta}$ sufficiently large, one also has, by Assumption A.1, $p(X^*)X^*f(\bar{\delta}I_o,\bar{\delta}\ell_o)/\bar{\delta}\ell_o < 1$ and hence

$$\chi(I,\bar{\delta}\ell_o) < p(X^*)X^*f_2(I,\bar{\delta}\ell_o) - 1$$

$$< p(X^*)X^*f(I,\bar{\delta}\ell_o)/\bar{\delta}\ell_o - 1$$

$$< p(X^*)X^*f(\bar{\delta}I_o,\bar{\delta}\ell_o)/\bar{\delta}\ell_o - 1 < 0$$

if $\delta(I) < \bar{\delta}$. The existence of $\lambda(I)$ such that $\chi(I,\lambda(I)) = 0$ follows by the intermediate value theorem. Moreover, $\lambda(I) \in (\delta(I)\ell_o,\bar{\delta}\ell_o)$.

As in Case 1, one easily verifies that the contract $(I,\lambda(I),X^*,w(.),$ $X^*f(I,\lambda(I)),\bar{w}(.))$ with $w(y) = \bar{w}(y) = (1-I/p(X^*)X^*f(I,\lambda(I)))y$ for all y satisfies the constraints (5*), (6*), (7), and (8). The entrepreneur's expected payoff from this contract is again given by (A.2), with A=0. to complete the proof, it suffices to show that for some I>0, $\zeta(I)$ exceeds the best the entrepreneur can obtain with I=0.

Since $\lambda(I) \in (\delta(I)\ell_o,\bar{\delta}\ell_o]$ for all I>0, there exists $\hat{\ell} \in [0,\delta\ell_o]$ such that $\hat{\ell} = \lim_{k\to\infty} \lambda(I_k)$ for some sequence $\{I_k\}$ with $\lim_{k\to\infty} I_k = 0$. I claim that $\hat{\ell} \in \text{argmax}$ $[p(X^*)X^*f(0,\ell) - \ell]$, i.e. that $p(X^*)X^*f(0,\hat{\ell}) - \hat{\ell}$ is the maximum the entrepreneur can obtain when I=0. By the definition of $\lambda(.)$, for any k, one has

$$\lambda(I_k) = \underset{\ell}{\operatorname{argmax}} \left\{ \left[ p(X^*)X^* - \frac{I_k}{f(I_k,\lambda(I_k))} \right] f(I,\ell) - \ell \right\}.$$

Since $\lambda(I_k) \geq \delta(I_k)\ell_o$, one also has $0 < I_k/f(I_k,\lambda(I_k)) < I_k/f(I_k,\delta(I_k)\ell_o) < 1/f_1(\delta(I_k)I_k,\delta(I_k)\ell_o)$. By Assumption A.1, it follows that $\lim_{k\to\infty} I_k/f(I_k,\lambda(I_k)) = 0$, and the claim that $\hat{\ell} \in \underset{\ell}{\operatorname{argmax}}\, [p(X^*)X^*f(0,\ell) - \ell]$ follows from the maximum theorem.

Suppose first that $\hat{\ell} = 0$. Then one has, for any k,

$$\begin{aligned}
\zeta(I_k) &= \max_{\ell} \left\{ \left[ p(X^*)X^* - \frac{I_k}{f(I_k,\lambda(I_k))} \right] f(I_k,\ell) - \ell \right\} \\
&\geq \max_{\ell} \left\{ \left[ p(X^*)X^* - \frac{I_k}{f(I_k,\delta(I_k)\ell_o)} \right] f(I_k,\ell) - \ell \right\} \\
&\geq p(X^*)X^*f(I_k,\delta(I_k)\ell_o) - I_k - \delta(I_k)\ell_o \\
&\geq p(X^*)X^*f(0,0) + [p(X^*)X^*f_1(I_k,\delta(I_k)\ell_o) - 1]\, I_k \\
&\qquad + [p(X^*)X^*f_2(I_k,\delta(I_k)\ell_o) - 1]\, \delta(I_k)\ell_o,
\end{aligned}$$

and Assumption A.1 implies $\zeta(I_k) > p(X^*)X^*f(0,0)$ for any sufficiently large k.

Alternatively, suppose that $\hat{\ell} > 0$. Then one has, for any k,

$$\begin{aligned}
\zeta(I_k) &\geq p(X^*)X^*f(0,\hat{\ell}) - \hat{\ell} + [p(X^*)X^*f_1(I_k,\hat{\ell}) - 1]\, I_k \\
&\qquad + [p(X^*)X^*f_2(I_k,\lambda(I_k)) - 1]\, (\lambda(I_k) - \hat{\ell}).
\end{aligned}$$

By the definition of $\lambda(I_k)$, it follows that

$$\begin{aligned}
\zeta(I_k) &\geq p(X^*)X^*f(0,\hat{\ell}) - \hat{\ell} + [p(X^*)X^*f_1(I_k,\hat{\ell}) - 1]\, I_k \\
&\qquad + \frac{I_k}{f(I_k,\lambda(I_k))} f_2(I_k,\lambda(I_k))(\lambda(I_k) - \hat{\ell}) \\
&\geq p(X^*)X^*f(0,\hat{\ell}) - \hat{\ell} + \left[ p(X^*)X^*f_1(I_k,\hat{\ell}) - 1 - \frac{f_2(I_k,\lambda(I_k))}{f(I_k,\lambda(I_k))}\, \bar{\delta}\ell_o \right] I_k
\end{aligned}$$

$$\geq p(X^*)X^*f(0,\hat{\ell}) - \hat{\ell} + \left[p(X^*)X^*f_1(I_k,\hat{\ell}) - 1 - \frac{\bar{\delta}\ell_o}{\lambda(I_k)}\right]I_k,$$

and Assumption A.1 implies $\zeta(I_k) > p(X^*)X^*f(0,\hat{\ell}) - \hat{\ell}$ for any sufficiently large k. Thus in either case, if $\hat{\ell}=0$ and if $\hat{\ell}>0$, for any sufficiently large k, the contract $(I_k,\lambda(I_k),X^*,w(.), X^*f(I_k,\lambda(I_k)),\bar{w}(.))$ with $w(y) = \bar{w}(y) = (1 - I_k)/p(X^*)X^*f(I_k,\lambda(I_k)))y$ for all y dominates the best that the entrepreneur can obtain with I=0. This completes the proof of Proposition 4.1.

*Q.E.D.*

PROOF of PROPOSITION 4.2

It suffices to show that the best contract with $\ell=0$ is dominated by some contract with $\ell>0$. Among contracts involving $\ell=0$, the entrepreneur can do no better than to set $I = \hat{I} \in \underset{I}{\operatorname{argmax}}\ p(X^*)X^*f(I,0) - I$, $\ell=0$, $X=X^*$, $\bar{y} = X^*f(\hat{I},0)$, and $w(y) = \bar{w}(y) = pX^*)X^*f(\hat{I},0) - \hat{I} + A$, regardless of y; this provides him with the payoff $\bar{w}(0) = p(X^*)X^*f(\hat{I},0) - \hat{I} + A$. If $\bar{w}(0) = 0$, it suffices to note that this is no better than the payoff from setting $I=\ell=0$, so by Proposition 1, $(\hat{I},0,X^*,w(.),\bar{y},\bar{w}(.))$ cannot be a solution to problem (4*). If $\bar{w}(0) > 0$, one can replace the contract $(\hat{I},0,X^*,w(.),\bar{y},\bar{w}(.))$ by a new contract $(\hat{I},\ell(\varepsilon),X^*,w_\varepsilon(.), \bar{y}(\varepsilon),\bar{w}_\varepsilon(.))$ where $\ell(\varepsilon) = \operatorname{argmax}\ [\varepsilon p(X^*)X^*f(\hat{I},\ell) - \ell]$, $\bar{y}(\varepsilon) = X^*f(\hat{I},\ell(\varepsilon))$, $w_\varepsilon(y) = \bar{w}_\varepsilon(y) = \varepsilon y + (1-\varepsilon)p(X^*)X^*f(\hat{I},\ell(\varepsilon)) - \hat{I} + A$ for any y, and $\varepsilon>0$ is sufficiently close to zero so that Assumption A.3 is not violated. This new contract provides the entrepreneur with the expected payoff $p(X^*)X^*f(\hat{I},>\ell(\varepsilon)) - \hat{I} - \ell(\varepsilon) + A$. Since Assumption A.1 implies $0 < \ell(\varepsilon) < \underset{\ell}{\operatorname{argmax}}\ [p(X^*)X^*f(\hat{I},\ell) - \ell]$, this payoff exceeds $p(X^*)X^*f(\hat{I},0) - \hat{I} + A$, the maximum payoff with $\ell=0$.

*Q.E.D.*

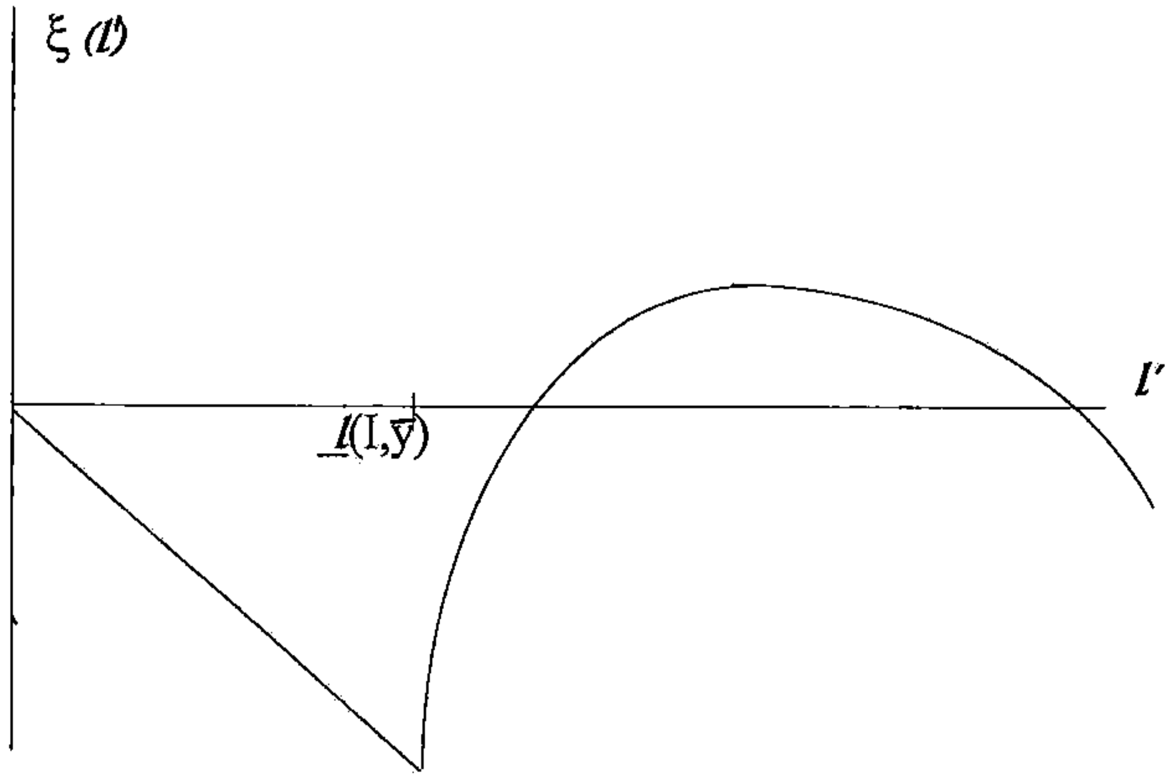
PROOF of LEMMA 4.3.

By Assumptions A.1 and A.2, the function

$$\ell' \to \xi(\ell') := p\left(\frac{\bar{y}}{f(I,\ell')}\right)[\bar{w}(\bar{y}) - \bar{w}(0)] - \ell'$$

is continuous. Define $\underline{\ell}(I,\bar{y})$ so that $\bar{X} = \bar{y}/f(I,\underline{\ell}(I,\bar{y}))$, where $\bar{X}$ is the critical risk level specified in Assumption A.2. For $\ell' \leq \underline{\ell}(I,\bar{y})$, Assumption A.2 yields $\xi(\ell') = -\ell'$. Above $\underline{\ell}(I,\bar{y})$, $\xi(.)$ is twice continuously

FIGURE 3



differentiable, with first derivative

$$\xi'(\ell') = -\,p'\left(\frac{\bar{y}}{f(I,\ell')}\right)\frac{\bar{y}}{f(I,\ell')}\,\frac{f_2(I,\ell')}{f(I,\ell')}\,[\bar{w}(\bar{y}) - \bar{w}(0)] - 1$$

and second derivative

$$\xi''(\ell') = \left[p''\left(\frac{\bar{y}}{f(I,\ell')}\right)\frac{\bar{y}}{f(I,\ell')} + 2p'\left(\frac{\bar{y}}{f(I,\ell')}\right)\right]\frac{\bar{y}}{f(I,\ell')}\left[\frac{f_2(I,\ell')}{f(I,\ell')}\right]^2[\bar{w}(\bar{y}) - \bar{w}(0)]$$

$$-\,p'\left(\frac{\bar{y}}{f(I,\ell')}\right)\frac{\bar{y}}{f(I,\ell')}\,\frac{f_{22}(I,\ell')}{f(I,\ell')}\,[\bar{w}(\bar{y}) - \bar{w}(0)],$$

which is negative, by Assumptions A.1 and A.2. As shown in Figure 3, it follows that $\xi(.)$ has at most two local maxima, one at $\ell' = 0$, and one at $\ell' > \underline{\ell}(I,\bar{y})$ satisfying $\xi'(\ell') = 0$. The first-order condition (11) is thus necessary and sufficient for $\ell > 0$ to yield a local maximum. This maximum is a global maximum if and only if it satisfies (12).

*Q.E.D.*

PROOF of LEMMA 5.1.
The Kuhn-Tucker condition for $\bar{w}(0)$ requires $\mu \geq 1$, with $\bar{w}(0) = 0$ if $\mu > 1$. Hence it suffices to show that one cannot have $\mu = 1$.

If $\mu$ were equal to one, $(I,\ell,X,\nu)$ would also be a solution to the Lagrangian problem

$$\underset{I,\ell,X}{\text{Max}}\ \underset{\nu\geq 0}{\text{Min}}\left[p(X)Xf(I,\ell) - \ell - I + A + \nu\left(r(X)\,\frac{f(I,\ell)}{f_2(I,\ell)} - \ell\right)\right].$$

However, this latter problem has a unique solution, namely the first-best outcome $(I^*,\ell^*,X^*)$ with $\nu=0$. (By (10c), $r(X^*) = 1$, and by Assumption A.1, $f(I^*,\ell^*)/f_2(I^*,\ell^*) - \ell^* > 0$, so $(I^*,\ell^*,X^*)$ satisfies (15) with a strict inequality.) Thus $\mu=1$ would imply that $I=I^*$, $\ell=\ell^*$, and $X=X^*$. However, by (10b) and (10c) in combination with Assumptions A.3 and A.4, one has

$$p(X^*)X^*f(I^*,\ell^*) - r(X)\,\frac{f(I^*,\ell^*)}{f_2(I^*,I^*)} - I^* + A - \bar{w}(0) = -\,I^* + A - \bar{w}(0) < 0,$$

which is incompatible with the Kuhn-Tucker condition for $\mu$. Therefore one cannot have $\mu = 1$.

*Q.E.D.*

PROOF of LEMMA 5.2.

By Proposition 4.1, the maximum value of the objective in problem (4*) and hence in problems (13) and (17) is strictly greater than $\max_{\ell'} [p(X^*)X^*f(A,\ell') - \ell']$, which is nonnegative. With $\bar{w}(0) = 0$, according to Lemma 5.1, it follows that any solution to problem (17) satisfies

$$r(X) \frac{f(I,\ell)}{f_2(I,\ell)} - \ell > 0$$

and hence, by the Kuhn-Tucker condition for $\nu$, $\nu=0$.

*Q.E.D.*

PROOF of PROPOSITION 5.3.

By a rearrangement of terms, (22) becomes:

$$\frac{p(X) + p'(X)X}{p(X)X}\left[p(X)Xf_2(I,\ell) - \frac{\mu-1}{\mu} r(X)\right] = \frac{\mu-1}{\mu} r(X) \quad \frac{(2p'(X)+p''(X)X)}{(-p'(X)X)}$$

Upon using (23) to substitute for $-r(X)$ in the second term on the left-hand side, one obtains:

$$\frac{p(X) + p'(X)X}{p(X)X} \left[\frac{1}{\mu} p(X)Xf_2(I,\ell) + \frac{\mu-1}{\mu} \frac{f_2(I,\ell)}{f(I,\ell)} (I-A)\right] = \frac{\mu-1}{\mu} \frac{(2p'(X)+p''(X)X)}{(-p'(X)X)}$$

Since Assumption A.2, Proposition 4.1, and Lemma 5.1 imply $2p''(X) + p''(X)X < 0$, $I>A$, and $\mu>1$, it follows that $p(X) + p'(X)X < 0$, and therefore $X>X^*$ and $r(X) < 1$.

*Q.E.D.*

PROOF of PROPOSITION 5.4.

Using (24), one can rewrite (29) as

$$p(X)Xg'(\psi(I,\ell))\psi(I,\ell) - (I+\ell) = \frac{\mu-1}{\mu} \left[W^* + r(X) \frac{g}{g'\psi_2} \delta\right].$$

As discussed in the proof of Lemma 5.2, $W^*>0$, so $\delta\geq 0$ implies

$$\text{(A.3)} \qquad p(X)Xg'(\psi(I,\ell))\psi(I,\ell) > I+\ell.$$

From the first-order conditions for $\hat{I}(X)$ and $\hat{\ell}(X)$, one also has

$$p(X)Xg'(\psi(\hat{I}(X),\hat{\ell}(X)))\psi_1(\hat{I}(X),\hat{\ell}(X)) = 1,$$

$$p(X)Xg'(\psi(\hat{I}(X),\hat{\ell}(X)))\psi_2(\hat{I}(X),\hat{\ell}(X)) = 1,$$

and hence, by Euler's theorem,

$$\text{(A.4)} \qquad p(X)Xg'(\psi(\hat{I}(X),\hat{\ell}(X)))\psi(\hat{I}(X),\hat{\ell}(X)) = \hat{I}(X) + \hat{\ell}(X).$$

By the definition of $(\hat{I}(X),\hat{\ell}(X))$, one also has

$$\frac{I(X) + \ell(X)}{\psi(\hat{I}(X),\hat{\ell}(X))} \leq \frac{\lambda I + \lambda \ell}{\psi(\lambda I,\lambda \ell)},$$

where $\lambda := (\hat{I}(X) + \hat{\ell}(X))/(I + \ell)$. Since $\psi$ is linearly homogeneous, it follows that

$$\text{(A.5)} \qquad \frac{I(X) + \ell(X)}{\psi(\hat{I}(X),\hat{\ell}(X))} \leq \frac{I + \ell}{\psi(I,\ell)}.$$

Upon combining (A.3), (A.4), and (A.5), one obtains $g'(\psi(\hat{I}(X),\hat{\ell}(X))) < g'(\psi(I,\ell))$, and hence

$$\psi(\hat{I}(X),\hat{\ell}(X)) > \psi(I,\ell).$$

By (24), it follows that $f(\hat{I}(X),\hat{\ell}(X)) > f(I,\ell)$. Moreover if $g'\psi$ is increrasing in $\psi$, then by (A.3) and (A.4), it follows that $\hat{I}(X) + \hat{\ell}(X) > I+\ell$.

*Q.E.D.*

PROOF of COROLLARY 5.5.
Since $p(X)X < p(X^*)X^*$, one has $f(\hat{I}(X),\hat{\ell}(X)) < f(I^*,\ell^*)$ and $\hat{I}(X) + \hat{\ell}(X) < I^* + \ell^*$. Hence the corollary follows immediately from Proposition 5.4.

*Q.E.D.*

PROOF of PROPOSITION 5.6.

Using (29), one can rewrite (30) as

$$\text{(A.6)} \qquad \left[I+\ell + \frac{\mu-1}{\mu} W^*\right]\frac{\psi_2-\psi_1}{\psi} = \frac{\mu-1}{\mu\ell}\left[W^* - r(X)\frac{g}{g'\psi_2}\frac{\sigma-1}{\sigma}\right].$$

Since $W^*>0$, $\sigma\leq1$ implies $\psi_2(I,\ell) > \psi_1(I,\ell)$. Since the first-order conditions for $\hat{I}(X)$ and $\hat{\ell}(X)$ imply $\psi_2(\hat{I}(X),\hat{\ell}(X)) = \psi_1(\hat{I}(X),\hat{\ell}(X))$, it follows that

$$\frac{\psi_2(I,\ell)}{\psi_1(I,\ell)} > \frac{\psi_2(\hat{I}(X),\hat{\ell}(X))}{\psi_1(\hat{I}(X),\hat{\ell}(X))}.$$

Since $\psi$ is concave and linearly homogeneous, $\psi_2/\psi_1$ is increasing in the investment/effort ratio. Therefore $\sigma\leq1$ implies $I/\ell > \hat{I}(X)/\hat{\ell}(X)$. For the case where $\sigma$ is large, use (28) to rewrite (A.6) in the form

$$\text{(A.6')} \qquad \left[I+\ell + \frac{\mu-1}{\mu} W^*\right]\frac{\psi_2-\psi_1}{\psi} = \frac{\mu-1}{\mu\ell}\left[r(X)\frac{g}{g'\psi_2}\frac{1}{\sigma} - 1\right].$$

and note that the right-hand side of (A.6') is negative if $\sigma$ is sufficiently large. In this case $\psi_2(I,\ell) < \psi_1(I,\ell)$, hence

$$\frac{\psi_2(I,\ell)}{\psi_1(I,\ell)} < \frac{\psi_2(\hat{I}(X),\hat{\ell}(X))}{\psi_1(\hat{I}(X),\hat{\ell}(X))},$$

and one must have $I/\ell < \hat{I}(X)/\hat{\ell}(X)$.

*Q.E.D.*

PROOF of COROLLARY 5.7.

By Proposition 5.4, $\delta\geq0$ implies $\ell < \hat{\ell}(X)$ or $I < \hat{I}(X)$ (or both). Thus $\ell \geq \ell(X)$ implies $I/\ell < \hat{I}(X)/\hat{\ell}(X)$ and, by Proposition 5.6, $\sigma>1$. Moreover $I \geq \hat{I}(X)$ implies $I/\ell > \hat{I}(X)/\hat{\ell}(X)$, in which case, by Proposition 5.6, $\sigma$ cannot be arbitrarily large.

*Q.E.D.*

PROOF of PROPOSITION 5.8.

As discussed in the text, the first-best outcome $(I^*,\ell^*,X^*)$ is independent of $\sigma$, with $I^* = aC^*$, $\ell^* = bC^*$, and $C^* = g'^{-1}(1/p(X^*)X^*)$. The overall expected surplus from this first-best outcome is equal to $W^{**} := p(X^*)X^*g(C^*) - C^*$, regardless of $\sigma$. Clearly $W^*(\sigma) < W^{**}$ for any $\sigma$.

For any $\sigma$, consider the outcome $(0,\underline{\ell}(\sigma),X^*)$ where $\underline{\ell}(\sigma) := \arg\max_{\ell'} [p(X^*)X^* g(b^{1/(\sigma-1)}\ell) - \ell]$. For $\sigma>1$, the contract $(0,\underline{\ell}(\sigma),X^*,w(.),\bar{y}',\bar{w}(.))$ with $\bar{y}' = X^*g(b^{1/(\sigma-1)}\underline{\ell}(\sigma))$ and $w(y) = \bar{w}(y) = A + y$ for all $y$ satisfies the constraints of problem (4*) and yields the payoff expectation $\underline{W}(\sigma) := p(X^*)X^*g(b^{1/(\sigma-1)}\underline{\ell}(\sigma))-\underline{\ell}(\sigma)$ to the entrepreneur. Clearly $W^*(\sigma) \geq \underline{W}(\sigma)$ for all $\sigma>1$.

Notice also that $\lim_{\sigma\to\infty} \underline{W}(\sigma) = W^{**}$. Since $W^*(\sigma) \in [\underline{W}(\sigma),W^{**})$ for all $\sigma>1$, it follows that $\lim_{\sigma\to\infty} W^*(\sigma) = W^{**}$. This immediately implies $\lim_{\sigma\to\infty} X(\sigma) = X^*$ and $\lim_{\sigma\to\infty} [I(\sigma) + \ell(\sigma)] = I^* + \ell^*$.

The hard part of the proof is to show that $I(\sigma)$ converges to A and $\ell(\sigma)$ converges to $I^* + \ell^* - A$, i.e. that the second-best contracts do *not* involve $(I(\sigma),\ell(\sigma),X(\sigma))$ converging to $(I^*+\ell^*,0,X^*)$, which actually would be an incentive-compatible outcome for the limiting case with $f(I,\ell) = g(I+\ell)$. For this purpose I first show that $I(\sigma)/\ell(\sigma) \leq a/b$ for any sufficiently large $\sigma$. Note that for given $\sigma$ and $X(\sigma)$, the investment effort combination $(I(\sigma),\ell(\sigma))$ must be minimizing the "cost"

$$\gamma(I,\ell) = I + \ell + (\mu-1)\left[r(X(\sigma))\,\frac{f(I,\ell)}{f_2(I,\ell)} + I\right]$$

of "producing" the "output" $f(I,\ell) = f(I(\sigma),\ell(\sigma))$ in (19). Thus at the point $(I(\sigma),\ell(\sigma))$ in $(I,\ell)$-space, any movement along the "output isoquant" $f(I,\ell) = f(I(\sigma),\ell(\sigma))$ must raise the "cost" $\gamma(I,\ell)$. At $(I(\sigma),\ell(\sigma))$ the "cost isoquant" $\gamma(I,\ell) = \gamma(I(\sigma),\ell(\sigma))$ must therefore have (i) the same slope as and (ii) a smaller curvature than the "output isoquant" $f(I,\ell) = f(I(\sigma),\ell(\sigma))$. Formally, one must have:[1]

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1 For $\psi$ satisfying (31a)-(31b), (A.7) follows from the first-order conditions (25) and (26) if one multiplies (25) by $\eta(\sigma)^{1/\sigma} = [bI(\sigma)/a\ell(\sigma)]^{1/\sigma}$ and subtracts the result from (26). Similarly, (A.8)

(A.7) $h(\eta(\sigma),\sigma) = 0$

and

(A.8) $h_1(\eta(\sigma),\sigma) \geq 0$,

where

(A.9) $$\eta(\sigma) := \frac{I(\sigma)/a}{\ell(\sigma)/b} ,$$

and, for any $\eta>0$,

(A.10) $$h(\eta,\sigma) = \eta^{1/\sigma} - 1 - \frac{\mu-1}{\mu}\left[r(X(\sigma))\,\frac{a\eta^{(\sigma-1)/\sigma} + b}{\sigma\beta(\sigma)b} - 1\right] ,$$

(A.11) $$\beta(\sigma) := \frac{g'(\bar{\psi}(I(\sigma),\ell(\sigma),(\sigma-1/\sigma))\ \bar{\psi}(I(\sigma),\ell(\sigma),(\sigma-1)/\sigma)}{g(\psi(I(\sigma),\ell(\sigma),(\sigma-1)/\sigma))}$$

For any $\sigma$, one has:

$$h(0,\sigma) = -\frac{1}{\mu} - \frac{\mu-1}{\mu}\, r(X(\sigma))/\ \sigma\beta(\sigma) < 0$$

and

$$h(1,\sigma) = \frac{\mu-1}{\mu}\left[1 - r(X(\sigma))\ /\ \sigma\beta(\sigma)b\right].$$

Since $r(X(\sigma))$ converges to one and $\beta(\sigma)$ converges to $g'(C^*)C^*/g(C^*)$ as $\sigma$ goes out of bounds, one has $h(1,\sigma) > 0$ for any sufficiently large $\sigma$. It follows that for any sufficiently large $\sigma$, there exists $\hat{\eta}(\sigma) \in (0,1)$ such that $h(\hat{\eta}(\sigma),\sigma) = 0$ and $h_1(\hat{\eta}(\sigma),\sigma) \geq 0$.

To prove that $\eta(\sigma) < 1$, it suffices to show that $\eta(\sigma) = \hat{\eta}(\sigma)$. Since $\eta(\sigma)$ must satisfy (A.7) and (A.8), this conclusion follows automatically if one finds that $\hat{\eta}(\sigma)$ is in fact the *unique* solution to the conditions $h(\eta,\sigma) = 0$ and $h_1(\eta,\sigma) \geq 0$. To prove this, note that

$$h_1(\eta,\sigma) = \frac{\eta^{(1-\sigma)/\sigma}}{\sigma}\left[1 - \frac{\mu-1}{\mu}\, r(X(\sigma))\,\frac{a\eta^{(\sigma-2)/\sigma}}{\beta(\sigma)b}\right]$$

so for $\sigma>2$, $h_1(\eta',\sigma) \geq 0$ for some $\eta'$ implies $h_1(\eta'',\sigma) < 0$ for all $\eta'' > \eta'$. Hence if $h(\eta_1,\sigma) = h(\eta_2,\sigma)$ for $\sigma>2$ and $\eta_1,\eta_2$ satisfying $\eta_1<\eta_2$, then $h_1(\eta_1,\sigma) >$

---

follows from the second-order conditions for problem (19).

0, $h_1(\eta_2,\sigma) < 0$, and indeed $h_1(\eta",\sigma) < 0$ for all $\eta" > \eta_2$. Thus for $\sigma>2$, the conditions $h(\eta,\sigma) = 0$ and $h_1(\eta,\sigma) \geq 0$ have no more than one solution.[2] It follows that for any sufficiently large $\sigma$, $h(\eta,\sigma) = 0$ and $h_1(\eta,\sigma) \geq 0$ imply $\eta = \hat{\eta}(\sigma)$; hence for any sufficiently large $\sigma$, $\eta(\sigma) = \hat{\eta}(\sigma) < 1$.

To complete the proof that $\lim_{\sigma\to\infty} I(\sigma) = A$, note that for f satisfying (24), (31a), and (31b), the financiers' participation constraint (23) becomes

$$\text{(A.12)} \qquad p(X(\sigma))X(\sigma)g(\psi(I(\sigma),\ell(\sigma),(\sigma-1)/\sigma))$$

$$- r(X(\sigma)) \frac{a\eta^{(\sigma-1)/\sigma}+b}{\beta(\sigma)b} \ell(\sigma) - I(\sigma) + A = 0 ,$$

where $\beta(\sigma)$ is again given by (A.11). Since $\eta(\sigma) \leq 1$ for any sufficiently large $\sigma$, one has

$$\lim_{\sigma\to\infty} \frac{a\eta^{(\sigma-1)/\sigma}+b}{a\eta(\sigma)+b} = 1$$

Since $\lim_{\sigma\to\infty} X(\sigma) = X^*$ and $\lim_{\sigma\to\infty} (I(\sigma) + \ell(\sigma)) = C^*$, one also has:

$$\lim_{\sigma\to\infty} p(X(\sigma))X(\sigma)g(\bar{\psi}(I(\sigma),\ell(\sigma),(\sigma-1)/\sigma)) = p(X^*)X^*g(C^*),$$

$$\lim_{\sigma\to\infty} r(X(\sigma)) = 1 ,$$

$$\lim_{\sigma\to\infty} \frac{a\eta(\sigma)+b}{b} \ell(\sigma) = C^* ,$$

$$\lim_{\sigma\to\infty} \beta(\sigma) = \frac{g'(C^*)C^*}{g(C^*)} ,$$

and hence

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2 For $\sigma>2$, one also has $\lim_{\eta=\infty} h(\eta,\sigma) = -\infty$. If $h(1,\sigma) > 0$, it follows that the condition $h(\eta,\sigma) = 0$ <u>must</u> also have a solution $\eta>1$, which however satisfies $h_1(\eta,\sigma) < 0$; this solution corresponds to a local minimum for problem (19).

$$\lim_{\sigma\to\infty}\left[p(X(\sigma))X(\sigma)g(\bar{\psi}(I(\sigma),\ell(\sigma),(\sigma-1)/\sigma)) - r(X(\sigma))\,\frac{a\eta^{(\sigma-1)/\sigma}+b}{b\beta(\sigma)}\,\ell(\sigma)\right]$$

$$= p(X^*)X^*g(C^*) - \frac{g(C^*)}{g'(C^*)} = 0.$$

Thus (A.12) implies $\lim_{\sigma\to\infty}(I(\sigma) = A$. Since $\lim_{\sigma\to\infty}\ell(\sigma) = C^* - \lim_{\sigma\to\infty}I(\sigma)$, this in turn implies $\lim_{\sigma\to\infty}\ell(\sigma) = I^*+\ell^*-A$. The remaining claims of Proposition 5.8 are then trivial.

*Q.E.D.*

PROOF of PROPOSITION 5.9.

For any $\varepsilon>0$, consider the surplus maximization problem:

$$\max_{C}\ [p(X^*)X^*g(C) - (1+\varepsilon a)C].$$

Let $\hat{C}(\varepsilon)$ and $\underline{W}(\varepsilon)$ be the maximizer and the maximum value of the objective function in this problem. I claim that $W^*(\sigma) \geq \underline{W}(\varepsilon)$ for any $\varepsilon>0$ and any sufficiently small $\sigma>0$. To establish this claim, I show that for any $\varepsilon>0$ and any sufficiently small $\sigma>0$, a contract implementing the outcome $((1+\varepsilon)a\hat{C}(\varepsilon), b\hat{C}(\varepsilon), X^*)$ satisfies the financiers' participation constraint. The inequality $W^*(\sigma)\geq\underline{W}(\varepsilon)$ then follows from the observation that the expected surplus from this outcome is

$$p(X^*)X^*g(\hat{C}(\varepsilon)\bar{\psi}((1+\varepsilon)a,b,(\sigma-1)/\sigma)) - (1+\varepsilon a)\hat{C}(\varepsilon),$$

which is no less then $\underline{W}(\varepsilon)$ because $\bar{\psi}((1+\varepsilon)a,b,(\sigma-1)/\sigma) > \bar{\psi}(a,b,(\sigma-1)/\sigma) = 1$.

With $\bar{w}(0) = 0$ and $\bar{w}(\bar{y})$ determined by incentive compatibility, the financiers' payoff expectation from the outcome $((1+\varepsilon)a\hat{C}(\varepsilon), b\hat{C}(\varepsilon), X^*)$ is

$$p(X^*)X^*g(\hat{C}(\varepsilon)\bar{\psi}((1+\varepsilon)a,b,(\sigma-1)/\sigma)) - (1+\varepsilon)a\hat{C}(\varepsilon) + A$$
$$- \frac{g(\hat{C}(\varepsilon)\bar{\psi}((1+\varepsilon)a,b,(\sigma-1)/\sigma))}{g'(\hat{C}(\varepsilon)\bar{\psi}((1+\varepsilon)a,b,(\sigma-1)/\sigma))}\ \frac{a(1+\varepsilon)^{1-1/\sigma}+b}{\bar{\psi}((1+\varepsilon)a,b,(\sigma-1)/\sigma)}$$

Note that $\lim_{\sigma\to 0}\bar{\psi}((1+\varepsilon)a,b,(\sigma-1)/\sigma) = 1$ and $\lim_{\sigma\to 0}(1+\varepsilon)^{1-1/\sigma} = 0$. As $\sigma$ converges to zero, the financiers' payoff expectation from the outcome $((1+\varepsilon)a,b,X^*)$ must therefore converge to

$$p(X^*)X^*g(\hat{C}(\varepsilon)) - (1+\varepsilon)a\hat{C}(\varepsilon) + A - \frac{g(\hat{C}(\varepsilon))}{g'(\hat{C}(\varepsilon))} b.$$

By the first-order condition for $\hat{C}(\varepsilon)$, this is equal to

$$p(X^*)X^*g(\hat{C}(\varepsilon)) \left[1 - \frac{b}{1+a\varepsilon}\right] - (1+\varepsilon)a\hat{C}(\varepsilon) + A$$

$$= \frac{(1+\varepsilon)a}{1+a\varepsilon} \left[p(X^*)X^*g(\hat{C}(\varepsilon)) - (1+a\varepsilon)\hat{C}(\varepsilon)\right] + A$$

$$= \frac{(1+\varepsilon)a}{1+a\varepsilon} \underline{W}(\varepsilon) + A,$$

which is strictly positive since Assumption A.1 implies $\underline{W}(\varepsilon) > 0$. It follows that for any $\varepsilon>0$ there exists $\bar{\sigma}(\varepsilon) > 0$ such that for any $\sigma \in (0,\bar{\sigma}(\varepsilon))$, the financiers' expected payoff from a contract implementing the outcome $((1+\varepsilon)a\hat{C}(\varepsilon), b\hat{C}(\varepsilon), X^*)$ is nonnegative.

Notice that $\lim_{\varepsilon\to 0} \underline{W}(\varepsilon) = W^{**}$ where $W^{**}$ is again the expected surplus from the first-best outcome $(I^*,\ell^*,X^*)$. Since $W^*(\sigma) < W^{**}$ for all $\sigma$ and $W^*(\sigma) > \underline{W}(\varepsilon)$ for all $\sigma \in (0,\bar{\sigma}(\varepsilon))$, it follows that $\lim_{\sigma\to 0} W^*(\sigma) = W^{**}$. This in turn implies $\lim_{\sigma\to 0} (I(\sigma),\ell(\sigma),X(\sigma)) = (I^*,\ell^*,X^*)$, and the remaining claims of Proposition 5.9 are trivial.

*Q.E.D.*

PROOF of PROPOSITION 6.1.

The proof proceeds in two steps. In the first step, I show that for any $w(.) \in \mathcal{W}$ and F satisfying (33), the resulting function $\bar{w}(.)$ satisfies (34a) everywhere. For any $w(.) \in \mathcal{W}$ and $\bar{y} > 0$, (8) and (33) yield

$$\bar{w}(\bar{y}) = \int_0^{\bar{\theta}} \frac{kw(\theta\bar{y})}{(k+1)\bar{\theta}}\, d\theta + \int_{\bar{\theta}}^{\infty} \frac{kw(\theta\bar{y})}{(k+1)\bar{\theta}} (\theta/\bar{\theta})^{-k-1}\, d\theta.$$

By a simple change of variables, this becomes

$$\bar{w}(\bar{y}) = \int_0^{\bar{\theta}\bar{y}} \frac{kw(x)}{(k+1)\bar{\theta}\bar{y}}\, dx + \int_{\bar{\theta}\bar{y}}^{\infty} \frac{kw(x)}{(k+1)\bar{\theta}\bar{y}} (x/\bar{\theta}\bar{y})^{-k-1}\, dx.$$

Since the density of F is continuous, one obtains

$$\frac{d\bar{w}}{d\bar{y}} = -\frac{1}{\bar{y}^2}\int_0^{\bar{\theta}\bar{y}} \frac{kw(x)}{(k+1)\bar{\theta}}\,dx + k\bar{y}^{-k-1}\int_{\bar{\theta}\bar{y}}^{\infty} \frac{kw(x)}{(k+1)\bar{\theta}}(x/\bar{\theta})^{-k-1}\,dx$$

$$= \frac{1}{\bar{y}}\left[k\bar{w}(\bar{y}) - (k+1)\int_0^{\bar{\theta}\bar{y}} \frac{kw(x)}{(k+1)\bar{\theta}\bar{y}}\,dx\right] \le k\,\frac{\bar{w}(\bar{y})}{\bar{y}},$$

where the last inequality follows from the fact that $w(.) \in \mathcal{W}$ is nonnegative-valued. This completes the first step of the proof.

In the second step, I prove (34b). By inspection of (9), since $U^*(\hat{y},\hat{w},\hat{I}) > 0$, there exists an open neighborhood $\mathcal{N}$ of $(\hat{y},\hat{w})$ such that $U^*(\bar{y},\bar{w},\hat{I}) > 0$ for all $(\bar{y},\bar{w}) \in \mathcal{N}$. By the argument in the proof of Lemma 4.3, it follows that there exists a function $\hat{\ell} : \mathcal{N} \to \mathbb{R}_{++}$ such that for any $(\bar{y},\bar{w}) \in \mathcal{N}$, $\hat{\ell}(\bar{y},\bar{w})$ is the unique global maximizer of the function $\ell' \to p(\bar{y}/f(\hat{I},\ell'))\bar{w} - \ell'$. By the maximum theorem, the function $\hat{\ell}$ is continuous on $\mathcal{N}$. By Assumptions A.1, A.2, and the envelope theorem, it follows that $U^*(.,.,\hat{I})$ is continuous differentiable on $\mathcal{N}$, with partial derivatives

$$U^*_{\bar{y}}(\bar{y},\bar{w},\hat{I}) = p'(\hat{X}(\bar{y},\bar{w}))\bar{w} \,/\, f(\hat{I},\hat{\ell}(\bar{y},\bar{w})) < 0$$

and

$$U^*_{\bar{w}}(\bar{y},\bar{w},\hat{I}) = p(\hat{X}(\bar{y},\bar{w})),$$

where $\hat{X}(\bar{y},\bar{w}) = \bar{y}/f(\hat{I},\hat{\ell}(\bar{y},\bar{w}))$. By the implicit function theorem, it follows that there exists a continuously differentiable function $\hat{w}$, with graph $\hat{w} \subset \mathcal{N}$ such that for all $(\bar{y},\bar{w}) \in \mathcal{N}$, $U^*(\bar{y},\bar{w},\hat{I}) \gtreqless U^*(\hat{y},\hat{w},\hat{I})$ as $\bar{w} \gtreqless \hat{w}(\bar{y})$; moreover the slope of $\hat{w}$ is, for any $\bar{y}$,

$$\text{(A.15)}\qquad \frac{d\hat{w}}{d\bar{y}}(\bar{y}) = -\frac{p'(\hat{X}(\bar{y},\hat{w}(\bar{y})))\hat{w}(\bar{y})}{p(\hat{X}(\bar{y},\hat{w}(\bar{y})))f(\hat{I},\hat{\ell}(\bar{y},\hat{w}(\bar{y})))} = \frac{1}{r(\hat{X}(\bar{y},\hat{w}(\bar{y})))}\,\frac{\hat{w}(\bar{y})}{\bar{y}}.$$

By construction, $\bar{y} = \hat{y}$ entails $\hat{w}(\bar{y}) = \hat{w}$ and $\hat{X}(\bar{y},\hat{w}(\bar{y})) = \hat{X}$, so (34b) follows immediately. As discussed in the text, Proposition 6.1 then follows because with $k < 1/r(\hat{X})$, (34a) and (34b) together imply

$$\frac{\hat{y}}{\bar{w}(\hat{y})}\,\frac{d\bar{w}}{d\bar{y}} < \frac{\bar{y}}{\hat{w}(\hat{y})}\,\frac{d\hat{w}}{d\bar{y}}(\hat{y}),$$

regardless of what underlying incentive scheme $\hat{w}(.) \in W$ is used.

*Q.E.D.*

PROOF of LEMMA 6.3.

For $r(\hat{X}) \geq 1$, it suffices to set $R = \hat{y}(1-r(\hat{X}))$ and $\alpha = 1-\hat{w}/r(\hat{X})\hat{y}$. In the case $r(\hat{X}) < 1$, consider a function $\nu(.)$ defined by the formula

$$\nu(z) = \frac{z}{r(\hat{X})} - \int_0^z \frac{\hat{\theta}(e^t)}{\hat{\theta}(e^t) - e^t}\, dt .$$

Note that $\hat{\theta}(e^t) \geq e^t$ for all t. Indeed one easily checks that if $F(e^z) < 1$, then for $t \leq z$, $\hat{\theta}(e^t) - e^t$ is bounded away from zero. Therefore $\nu(.)$ is well defined and continuous at any z satisfying $F(e^z) < 1$.

Since $\hat{\theta}(.)$ is nondecreasing and $\hat{\theta}(0) = 1$, one has $\hat{\theta}(e^t) \geq 1$ for all t. For $t < \ln(1-r(\hat{X}))$, it follows that $\hat{\theta}(e^t)/(\hat{\theta}(e^t) - e^t) \leq 1/(1-e^t) < 1/r(\hat{X})$. Hence $\nu(z) < \nu(\ln(1-r(\hat{X})))$ for all $z < \ln(1-r(\hat{X}))$.

Since $\lim_{\rho\to\infty} \hat{\theta}(\rho)/\rho = 1$, there exists $\bar{z}$ such that for all $t > \bar{z}$, $\hat{\theta}(e^t)/e^t \leq 1 + r(\hat{X})$, and hence $\hat{\theta}(e^t)/(\hat{\theta}(e^t) - e^t) \geq 1 + 1/r(\hat{X})$. For any $z > \bar{z}$, it follows that $\nu(z) \leq \nu(\bar{z}) + [1/r(\hat{X}) - (1+1/r(\hat{X}))](z-\bar{z}) < \nu(\bar{z})$.

By continuity, the restriction of $\nu(.)$ to the compact interval $[\ln(1-r(\hat{X})),\bar{z}]$ has a maximum, say at $z^*$. Since $\nu(\ln(1-r(\hat{X}))) > \nu(z)$ for all $z < \ln(1-r(\hat{X}))$ and $\nu(\bar{z}) > \nu(z)$ for all $z > \bar{z}$, $z^*$ is in fact a global maximizer of $\nu(.)$ on $\mathbb{R}$. It follows that

$$\text{(A.16)} \qquad \int_z^{z^*} \frac{\hat{\theta}(e^t)}{\hat{\theta}(e^t) - e^t}\, dt \leq \frac{z^*-z}{r(\hat{X})}$$

for all $z \in \mathbb{R}$.

I claim that $\hat{\theta}(.)$ and $F(.)$ are continuous at $e^{z^*}$, and that

$$\text{(A.17)} \qquad \frac{\hat{\theta}(e^{z^*})}{\hat{\theta}(e^{z^*})-e^{z^*}}\, dt = \frac{1}{r(\hat{X})} .$$

To see this, note that, by definition, $\hat{\theta}(.)$ is continuous from the left, so

(A.16) applied to a sequence $\{z^k\}$ which converges to $z^*$ from below yields

$$\frac{\hat\theta(e^{z^*})}{\hat\theta(e^{z^*})-e^{z^*}} \le \frac{1}{r(\hat X)} .$$

Further, since $\rho'>\rho$ implies $\hat\theta(\rho')/\rho' \ge \hat\theta(\rho)/\rho'$, the function $\rho \to \hat\theta(\rho)/\rho$ is lower semi-continuous from the right, and the function $t \to \hat\theta(e^t)/(\hat\theta(e^t) - e^t)$ is upper semi-continuous from the left. Applying (A.16) to a sequence $\{z^k\}$ which converges to $z^*$ from above, one therefore obtains

$$\frac{\hat\theta(e^{z^*})}{\hat\theta(e^{z^*})-e^{z^*}} \ge \lim_{k\to\infty} \sup_{t\in[z^*,z^k]} \frac{\hat\theta(e^t)}{\hat\theta(e^z)-e^t} \ge \frac{1}{r(\hat X)} .$$

Equation (A.17) follows immediately. Moreover, one must have

$$\frac{\hat\theta(e^{z^*})}{\hat\theta(e^{z^*})-e^{z^*}} \ge \lim_{k\to\infty} \sup_{t\in[z^*,z^k]} \frac{\hat\theta(e^t)}{\hat\theta(e^t)-e^t} .$$

This implies that $\hat\theta(.)$ and hence also $F(.)$ are continuous at $e^{z^*}$.

Now set $R = \hat y\, e^{z^*}$ and $\alpha = 1 - \hat w/\hat y(\hat\theta(e^{z^*}) - e^{z^*})(1-F(e^{z^*}))$. Then for any $\bar y > 0$,

$$\bar w(\bar y) = \frac{\hat w \int_{R/\bar y}^{\infty} (\theta\bar y - R)\, dF(\theta)}{\hat y \int_{R/\hat y}^{\infty} (\theta - R/\hat y)\, dF(\theta)} .$$

Clearly, $\bar w(\hat y) = \hat w$, so (32a) is satisfied. Moreover, for any $\bar y$,

$$\frac{d\bar w}{d\bar y} = \frac{\hat w \int_{R/\bar y}^{\infty} \theta\, dF(\theta)}{\hat y \int_{R/\bar y}^{\infty} (\theta - R/\hat y)\, dF(\theta)} = \frac{\bar w(\bar y)}{\bar y}\, \frac{\hat\theta(R/\bar y)}{\hat\theta(R/\bar y) - R/\bar y} ;$$

for $\bar y = \hat y$, this yields

$$\frac{d\bar{w}}{d\bar{y}}(\hat{y}) = \frac{\hat{w}}{\hat{y}} \frac{\hat{\theta}(e^{z^*})}{\hat{\theta}(e^{z^*})-e^{z^*}} ,$$

so (A.17) and (A.15) yield

$$\frac{d\bar{w}}{d\bar{y}}(\hat{y}) = \frac{d\hat{w}}{d\bar{y}}(\hat{y}) ,$$

confirming (32c). Finally, I note that for any $\bar{y}>0$, one has

$$\ln \bar{w}(\bar{y}) - \ln \bar{w}(\hat{y}) = \ln \int_{R/\bar{y}}^{\infty} (\theta\bar{y} - R)\, dF(\theta) - \ln \int_{R/\bar{y}}^{\infty} (\theta\hat{y} - R)\, dF(\theta)$$

$$= \int_{\hat{y}}^{\bar{y}} \frac{\int_{R/\bar{y}}^{\infty} \theta\, dF(\theta)}{\int_{R/\bar{y}}^{\infty} (\theta\bar{y} - \hat{R})dF(\theta)}\, d\bar{y}'$$

$$= \int_{\hat{y}}^{\bar{y}} \frac{\hat{\theta}(R/\bar{y}')}{\hat{\theta}(R/\bar{y}') - R/\bar{y}'}\, d\ln \bar{y}' = \int_{\ln R - \ln\bar{y}}^{\ln R - \ln\hat{y}} \frac{\hat{\theta}(e^t)}{\hat{\theta}(e^t) - e^t}\, dt ,$$

where the last equation involves the change of variables $t = \ln R - \ln \bar{y}'$. Now $\ln R - \ln \hat{y} = z^*$, so (A.16) implies

$$\ln \bar{w}(\bar{y}) - \ln \bar{w}(\hat{y}) \leq \frac{z^* - (\ln R - \ln \bar{y})}{r(\hat{X})} = \frac{\ln \bar{y} - \ln \hat{y}}{r(\hat{X})} ,$$

which proves (36).

*Q.E.D.*

PROOF of REMARK 6.4.

If F has a density $\varphi$, then by l'Hospital's rule, one finds

$$\lim_{\rho\to\infty} \frac{\hat{\theta}(\rho)}{\rho} = \lim_{\rho\to\infty} \frac{-\rho\varphi(\rho)}{1-F(\rho)-\rho\varphi(\rho)} = \lim_{\rho\to\infty} \frac{1}{1 - (1-F(\rho))/\rho\varphi(\rho)}$$

so $\lim_{\theta\to\infty} \theta\varphi(\theta)/(1-F(\theta)) = \infty$ implies $\lim_{\rho\to\infty} \hat{\theta}(\rho)/\rho = 1$.

*Q.E.D.*

PROOF of PROPOSITION 6.5.

Let $(I,\ell,X,\mu)$ be a solution to problem (19), and set $\hat{y} = Xf(I,\ell)$, $\hat{w} = f(I,\ell)/(-p'(X)Xf_2(I,\ell))$. In view of Lemmas 5.1 and 5.2, for any $w(.) \in W$ and $\bar{w}(.)$ satisfying $\bar{w}(\hat{y}) = \hat{w}$, the contract $(I,\ell,X,w(.),\hat{y},\bar{w}(.))$ maximizes the objective function (13) under the constraints (7), (11), (14), and (15), i.e., all the constraints of problem (13) *except* for (8) and (16). If $w(.) \in W$ and $\bar{w}(.)$ are designed so that (8) and (16) hold as well, it follows that $(I,\ell,X,w(.),\hat{y},\bar{w}(.))$ is a solution to problem (13) and hence, by Lemma 4.3, a solution to problem (4*).

The argument in the proof of Lemma 5.2 implies $U^*(\hat{y},\hat{w},I) > 0$. Moreover Proposition 5.3 implies $r(\hat{X}) < 1$. By Lemma 6.3 then, there exist constants $\alpha$ and R such that for w(.) defined by (35) and $\bar{w}(.)$ defined by (8), (32a), (32c), and (36) are satisfied. From the proof of Lemma 6.3, one sees that $R>0$ and $1-\alpha>0$. To complete the proof of Proposition 6.5, it suffices to show that the global incentive constraint (32b) is also satisfied so that with w(.) and $\bar{w}(.)$ given by the specified $\alpha$ and R, the contract $(I,\ell,X,w(.),\hat{y},\bar{w}(.))$ satisfies *all* the constraints of problem (13) and is indeed a solution to problem (13).

Consider the entrepreneur's indifference curve through the point $(\hat{y},\hat{w})$. Equation (A.15) shows that the elasticity of $\hat{w}(\bar{y})$ with respect to $\bar{y}$ is equal to $1/r(\hat{X}(\bar{y},\hat{w}(\bar{y})))$, so the question is how $\hat{X}(\bar{y},\hat{w}(\bar{y}))$ varies with $\bar{y}$. By standard calculations, one finds

$$\hat{X}_{\bar{w}}(\bar{y},\bar{w}) = \frac{1}{\dfrac{f_{22}f}{f_2^2} + \dfrac{p''(\hat{X})\hat{X} + 2p'(\hat{X})}{(-p'(\hat{X})\hat{X})}} \frac{\hat{X}}{\bar{w}} < 0$$

and

$$\hat{X}_{\bar{y}}(\bar{y},\bar{w}) = \frac{\dfrac{f_{22}f}{f_2^2} - 1}{\dfrac{f_{22}f}{f_2^2} + \dfrac{p''(\hat{X})\hat{X} + 2p'(\hat{X})}{(-p'(\hat{X})\hat{X})}} \frac{\hat{X}}{\bar{y}} > 0 ,$$

where in each case f, $f_2$, $f_{22}$ are evaluated at $(I,\hat{\ell}(\bar{y},\bar{w}))$. Along the indifference curve $\hat{r}(.)$, one then has

$$\frac{d\hat{X}(\bar{y},\hat{w}(\bar{y}))}{d\bar{y}} = \hat{X}_{\bar{y}} + \hat{X}_{\bar{w}} \frac{d\hat{w}}{d\bar{y}} = \frac{\dfrac{f_{22}f}{f_2^2} - 1 + \dfrac{1}{r(\hat{X})}}{\dfrac{f_{22}f}{f_2^2} + \dfrac{p''(\hat{X})\hat{X} + 2p'(\hat{X})}{(-p'(\hat{X})\hat{X})}} \frac{\hat{X}}{\bar{y}},$$

and hence,

$$\frac{d\hat{X}(\bar{y},\hat{w}(\bar{y}))}{d\bar{y}} \gtreqless 0 \quad \text{as} \quad r(\hat{X}) \left[1 - \frac{ff_{22}}{f_2^2}\right] - 1 \gtreqless 0.$$

As discussed in the context of conditions (21) and (26), if f is homothetic and takes the form (24), one has

$$r(\hat{X}) \left[1 - \frac{ff_{22}}{f_2^2}\right] - 1 = \frac{U^*}{\hat{\ell}} + r(\hat{X}) \frac{g(\psi)}{g'(\psi)\psi} \left[\delta - \frac{I\psi_1}{\hat{\ell}y_2} \frac{\sigma-1}{\sigma}\right],$$

where again all terms are evaluated at I and $\hat{\ell} = \hat{\ell}(\bar{y},\hat{w}(\bar{y}))$, respectively $\psi = \psi(I,\hat{\ell})$, $\psi_i = \psi_i(I,\hat{\ell})$, i=1,2, etc. since $U^*>0$ and, by assumptions $\delta$ and $1-\sigma$ are everywhere nonnegative, it follows that

(A.18) $$\frac{d\hat{X}(\bar{y},\hat{w}(\bar{y}))}{d\bar{y}} > 0$$

for all $\bar{y}$. Given Assumption A.2 and (A.15), the elasticity of $\hat{w}$ with respect to $\bar{y}$ must therefore be increasing in $\bar{y}$ whenever $\hat{X}(\bar{y},\hat{w}(\bar{y})) > X^*$. Moreover, since $r(X) < 1$, (A.18) implies that there exists $\underline{y} < \hat{y}$ such that $\hat{X}(\bar{y},\hat{w}(\bar{y})) > X^*$ if and only if $\bar{y} > \underline{y}$. For $\bar{y} > \underline{y}$, the monotonicity of the elasticity of $\hat{w}$ yields

(A.19) $$\bar{y} \frac{d \ln \hat{w}}{d\bar{y}}(\bar{y}) \gtreqless \hat{y} \frac{d \ln \hat{w}}{d\bar{y}}(\hat{y}) \quad \text{as } \bar{y} \gtreqless \hat{y}.$$

For $\bar{y} \leq \underline{y}$, $\hat{X}(\bar{y},\hat{w}(\bar{y})) \leq X^*$, so (A.15) implies

$$\bar{y} \frac{d \ln \hat{w}}{d\bar{w}} (\bar{y}) \leq 1 < \hat{y} \frac{d \ln \hat{w}}{d\bar{y}} (\hat{y}),$$

showing that (A.19) holds for $\bar{y} \leq \underline{y}$ as well as $\bar{y} > \underline{y}$.

Now (A.19) implies

$$\ln \hat{w}(\bar{y}) - \ln \hat{w}(\hat{y}) \geq [\ln \bar{y} - \ln \hat{y}]\, \hat{y}\, \frac{d\ \ln \hat{w}}{d\bar{w}}(\hat{y}) = [\ln \bar{y} - \ln \hat{y}]/r(\hat{X}).$$

Upon combining this with (36) and noting that $\hat{w}(\hat{y}) = \hat{w} = \bar{w}(\hat{y})$, one finds that indeed $\ln \hat{w}(\bar{y}) \geq \ln \bar{w}(\bar{y})$ for all $\bar{y}$, so the global incentive constraint (32b) is satisfied, and the contract $(I,\ell,X,w(.),\hat{y},\bar{w}(.))$ is indeed a solution to problem (13) and hence to problem (4*).

*Q.E.D.*

PROOF of REMARK 6.6.

Fix some $\varepsilon>0$ so that $\varepsilon<r(X)$ and suppose that

$$\text{Var}\ \tilde{\theta} < \frac{\varepsilon^2}{1-\varepsilon} \min \left[ r(X) - \varepsilon,\ 1 - 1/p(X)Xf_2(I,\ell) \right].$$

By Chebyshev's inequality, one then has

$$\text{(A.20)} \qquad F(1-\varepsilon) < \frac{1}{1-\varepsilon} \min \left[ r(X) - \varepsilon,\ 1 - 1/p(X)Xf_2(I,\ell) \right].$$

Since $\hat{\theta}(1-\varepsilon) \leq 1/(1-F(1-\varepsilon))$, it follows that $\hat{\theta}(1-\varepsilon)/(1-\varepsilon) < 1/(1-r(X))$, and, by the monotonicity assumption on $\hat{\theta}(\rho)/\rho$, $\hat{\theta}(\rho)/\rho < 1/(1-r(X))$ for all $\rho > 1-\varepsilon$. The "debt obligation" parameter R in Proposition 6.5 must then satisfy $R/Xf(I,\ell) < 1-\varepsilon$, so (38) yields

$$\begin{aligned} \alpha &\geq 1 - 1/p(X)Xf_2(X,\ell) \int_{1-\varepsilon}^{\infty} \theta\ dF(\theta) \\ &\geq 1 - 1/p(X)Xf_2(X,\ell)(1-(1-\varepsilon)F(1-\varepsilon)) \\ &> 0\ , \end{aligned}$$

the last inequality following by another application of (A.20).

*Q.E.D.*

PROOF of REMARK 6.7.

Note that the conditional-expectation function $\hat{\theta}(.)$ is strictly increasing on the support of F. Since $\hat{\theta}(0) = 1$, it follows that if $F(1-r(X)) > 0$, then $\hat{\theta}(1-r(X)) > 1$. Given that $\hat{\theta}(\rho)/\rho$ is assumed to be nonincreasing in $\rho$, this implies that $\hat{\theta}(\rho)/\rho > 1/(1-r(X))$ for all $\rho \le 1-r(X)$. The solution $\rho = R/Xf(I,\ell)$ to equation (37*) must therefore exceed 1-r(X); by equation (38), this implies:

$$\text{(A.21)} \qquad \alpha < 1 - 1/p(X)Xf_2(I,\ell) \int_{1-r(X)}^{\infty} \theta dF(\theta).$$

The monotonicity assumption on $\hat{\theta}(\rho)/\rho$ also yields

$$\int_{1-r(X)}^{\infty} \theta dF(\theta) \le \frac{1-r(X)}{\varepsilon} (1-F(1-r(X)))\, \hat{\theta}(\varepsilon)$$

$$\le \frac{1-r(X)}{\varepsilon} \left(1 - \frac{F(1-r(X)) - F(\varepsilon)}{1 - F(\varepsilon)}\right)$$

$$\le \frac{1-r(X)}{\varepsilon} (1 - (F(1-r(X)) - F(\varepsilon)))$$

for any $\varepsilon \in (0,1-r(X))$. Hence, if for some $\varepsilon>0$, $F(1-r(X)) - F(\varepsilon)$ is close to one, $\int_{1-r(X)}^{\infty} \theta dF(\theta)$ is close to zero, so, by (A.21), $\alpha$ is large and negative.

*Q.E.D.*

FIGURE 1

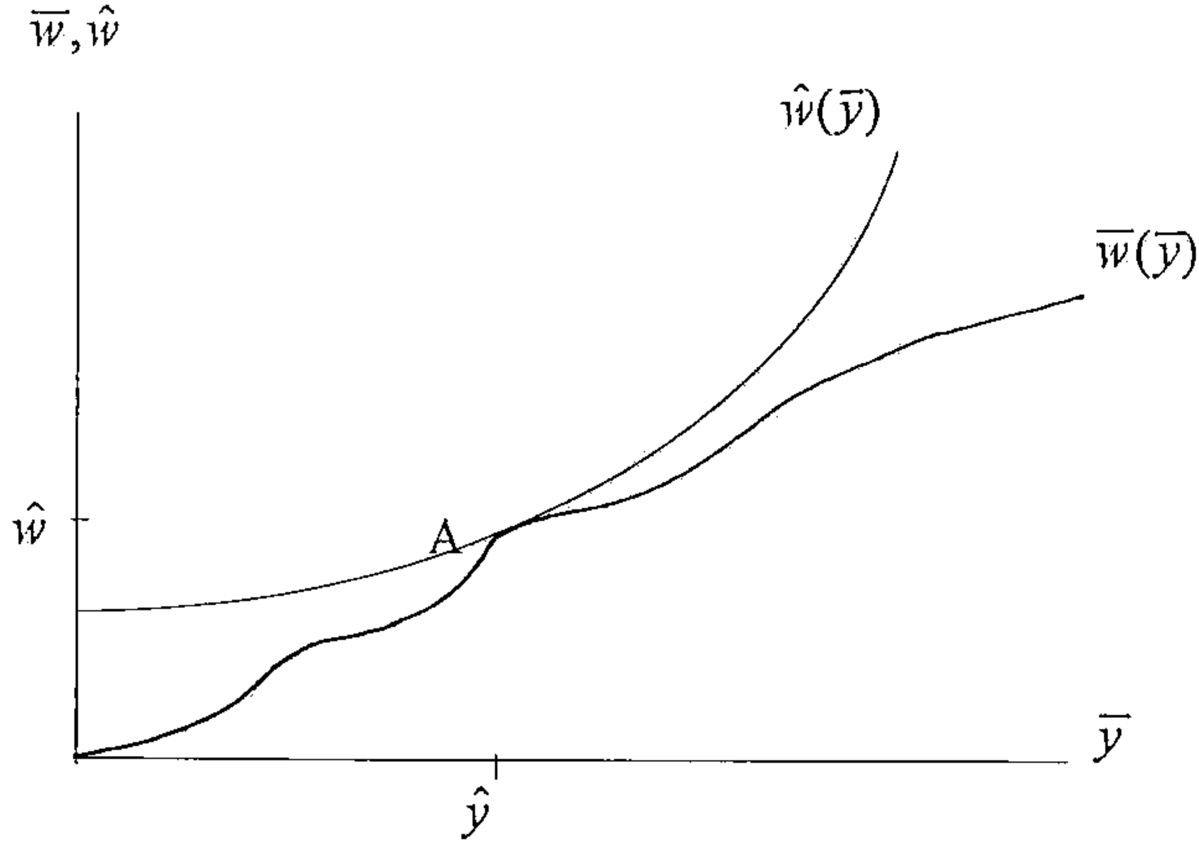
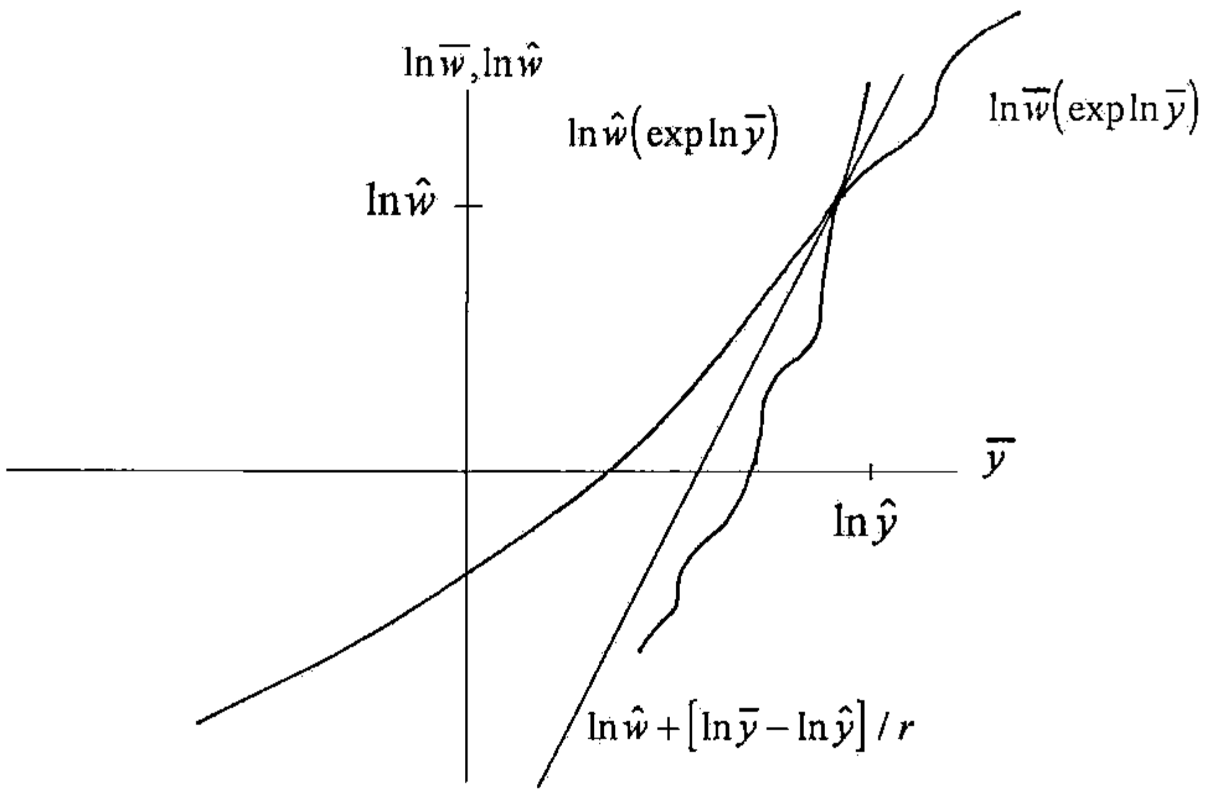

$\overline{w}, \hat{w}$
$\hat{w}(\overline{y})$
$\overline{w}(\overline{y})$
$\hat{w}$
A
$\overline{y}$
$\hat{y}$

FIGURE 2



$\ln\overline{w}, \ln\hat{w}$
$\ln\hat{w}(\exp\ln\overline{y})$
$\ln\overline{w}(\exp\ln\overline{y})$
$\ln\hat{w}$
$\overline{y}$
$\ln\hat{y}$
$\ln\hat{w} + [\ln\overline{y} - \ln\hat{y}] / r$

FIGURE 3