The Generalized Informativeness Principle*

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Abstract

The informativeness principle, as originally formulated by Holmstrom (1979), relies on the first-order approach. This paper extends the informativeness principle to settings in which the first-order approach cannot be applied. We introduce a "generalized informativeness principle" that takes into account non-local incentive constraints and holds generically. The principle provides a sufficient condition for a signal to have value when the first-order approach cannot be applied. This condition is also necessary for a signal to have value in the absence of restrictions on the utility function.

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1 Introduction

The informativeness principle, also known as the sufficient statistic theorem, states that all signals that are informative about agent effort should be included in a contract. This principle is believed to be the most robust result from the moral hazard literature. For example, Bolton and Dewatripont's (2005) textbook states that this literature has produced very few general results, but the informativeness principle is one of the few results that is general. Due to its perceived robustness, the principle has had substantial impact in several fields, such as compensation, insurance, and regulation. For example, in Bebchuk and Fried's (2004) influential book arguing that executive contracts are inefficient, one of their leading arguments is that contracts violate the informativeness principle by not taking into account peer performance.

The original formulation of the principle, in Holmstrom (1979) and Shavell (1979), assumes continuous effort and the validity of the first-order approach ("FOA"): that the agent's incentive constraint can be replaced by its first-order condition. As a consequence, only the likelihood ratio involving adjacent efforts is relevant. Thus, an informative signal is one that affects this "local likelihood ratio"; doing so is a necessary and sufficient condition for a signal to add value. All generalizations of the informativeness principle assume either the FOA (e.g. Gjesdal (1982), Amershi and Hughes (1989), Kim (1995)) or that the agent chooses between two actions only (e.g. Hart and Holmstrom (1987), Laffont and Martimort (2002), Bolton and Dewatripont (2005)). As is well-known, the FOA is generally not valid.[1] Assuming only two actions has a similar effect to using the FOA, as it means that only one incentive constraint binds, but is unrealistic.

The failure of the FOA is not simply a technical curiosity; there are many real-life situations where a single local incentive constraint does not ensure global incentive compatibility. Many agent decisions cannot be ordered, such as the choice of a corporate strategy, factory location, or whom to hire or promote. This is especially troublesome in multitask settings, where the agent can deviate in several different directions. Even with ordered actions, non-local deviations may bind if actions have increasing returns

[1] Rogerson (1985) derives the most well-known sufficient conditions for the validity of the FOA in the single-signal case. As Jewitt (1988) points out, these assumptions are so strong that they are not satisfied by any standard distribution. Moreover, they are no longer sufficient if the principal observes multiple signals, which is needed to analyze the informativeness principle (as the principal observes output and an additional signal). Sinclair-Desgagné (1994), Conlon (2009), and Kirkegaard (2015) obtain sufficient conditions for the validity of the FOA in the multiple-signal case.

to scale. For example, an academic who normally goes to the office on a weekday may contemplate working from home on that day, rather than only contemplating working one fewer minute in the office. The probability of discovering a blockbuster drug is likely convex in R&D effort (within some range): increasing effort from low to moderate has little effect on the probability, but increasing it from high to very high has a disproportionate impact.

Due to the significance of the informativeness principle and the restrictive setting in which it was derived, it is important to understand whether it holds more generally. As a preliminary step, we show that the informativeness principle may not hold when the FOA is invalid. A signal that affects the local likelihood ratio will have zero value if the agent is most likely to deviate to a non-local effort level.

Our main contribution is to introduce a "generalized informativeness principle" that provides a sufficient condition for a signal to have value when the FOA cannot be applied. This sufficient condition is stronger than in the original informativeness principle. Since we do not know ex ante which incentive constraint(s) will bind, the generalized informativeness principle postulates that a sufficient condition for a signal to be informative is that it affects the likelihood ratio between the principal's preferred effort and all other effort levels, rather than only adjacent effort levels.

Surprisingly, we show that even this generalized informativeness principle may not hold – a signal may affect all likelihood ratios yet still have zero value. Such violations arise if multiple incentive constraints bind, which is not uncommon (i.e., arises for an open set of parameters) when there are more than two effort levels. The principal can use the signal to transfer wages from states with low likelihood ratios to states with high likelihood ratios, thus relaxing one binding constraint. However, this transfer may tighten another binding constraint by the same magnitude. Such counter-examples are knife-edge in that they require the shadow prices of the binding constraints to exactly coincide. Accordingly, we prove that, except for a set of parameters with measure zero, any signal that affects the likelihood ratio for all other effort levels has positive value. Thus, the generalized informativeness principle holds generically.

We also show that the generalized informativeness principle contains the weakest sufficient condition for a signal to have value without making assumptions on the utility function. Thus, the principle also provides a necessary condition for a signal to have value without restricting the utility function (e.g. the cost of effort).[2]

[2]Our necessity result is in the spirit of the monotone comparative statics literature (see, e.g., Athey (2002)). Formally, it states that if the set of admissible utility functions is large enough to

2 Preliminaries

This is a preliminary section that defines what it means for a signal to have value, reviews the informativeness principle as originally formulated by Holmstrom (1979), and shows that it may be violated if the FOA is invalid. There is a risk-neutral principal ("she") and a risk-averse agent ("he"). The agent chooses an unobservable action $e \in \mathcal{E}$, which we refer to as "effort". Effort affects output $q \in \mathcal{Q}$ and a signal $s \in \mathcal{S}$, both of which are observable and contractible. We refer to a pair (q, s) as a "state." In Section 3, we will assume that the action, output, and signal spaces are finite; for now, to achieve comparability with Holmstrom (1979), we allow them to be intervals of the real line as well.

While Holmstrom (1979) assumes additive separability, we follow Grossman and Hart (1983) and generalize to the following utility function:

Assumption 1. *The agent's Bernoulli utility function over income w and effort e is*

$$U(w, e) = G(e) + K(e) V(w). \tag{1}$$

(i) $K(e) > 0$ *for all e;* **(ii)** $V : \mathcal{W} \to \mathbb{R}$ *is continuously differentiable, strictly increasing, and strictly concave, and $\mathcal{W} = (\underline{w}, +\infty)$ is an open interval of the real line (possibly with $\underline{w} = -\infty$); and* **(iii)** $U(w_1, e_1) \geq U(w_1, e_2) \implies U(w_2, e_1) \geq U(w_2, e_2)$ *for all $e_1, e_2 \in \mathcal{E}$ and all $w_1, w_2 \in \mathcal{W}$.*

The agent has utility function (1) if and only if his preferences over income lotteries are independent of his effort. Conditions (i) and (ii) state that the agent likes money and dislikes risk. Condition (iii) requires preferences over known effort levels to be independent of income. When $K(e) = \bar{K}$ for all e, the utility function is additively separable between effort and income as in Holmstrom (1979). When $G(e) = 0$ for all e, it is multiplicatively separable.[3] The agent's reservation utility is $\overline{U}$.

Note that we do not require effort to be ordered. Therefore, our model allows for the standard interpretation of e as effort (which unambiguously reduces utility and improves the output distribution) and more general cases where the action cannot be ordered (such as the choice of different corporate strategies or between multidimensional

include at least certain additively separable utility functions, then no weakening of the generalized informativeness principle can be obtained.

[3]Multiplicative separability is commonly used in macroeconomics (e.g. Cooley and Prescott (1995)). In finance, Edmans, Gabaix, and Landier (2009) show that they are necessary and sufficient to obtain empirically consistent scalings of CEO incentives with firm size.

tasks[4]). As an example of the former, effort e could refer to the number of hours worked. As an example of the latter, $e = 1$ could refer to working 8 hours on project A, $e = 2$ to working 9 hours on project A, and $e = 3$ to working 8 hours on project B.

As Grossman and Hart (1983) show, the principal's problem can be split in two stages. First, she finds the cheapest contract that induces each effort level. Second, she determines which effort level to induce. This paper focuses on the first stage: whether the principal can use the signal s to reduce the cost of implementing a given effort level.[5]

We first define what it means for a signal to have positive value. Let $\mathbb{E}_{(q,s)}[\cdot|e]$ denote the conditional expectation with respect to the distribution of states and $\mathbb{E}_q[\cdot|e]$ denote the conditional expectation with respect to the (marginal) distribution of outputs. When the principal uses the signal s, her cost of implementing effort e^* is

$$\mathcal{C}^s(e^*) \equiv \min_{w(q,s)} \mathbb{E}_{(q,s)}[w(q,s)|e^*] \tag{2}$$

subject to the agent's individual rationality ("IR") and incentive compatibility ("IC") constraints:

$$\mathbb{E}_{(q,s)}[U(w(q,s),e^*)|e^*] \geq \bar{U}, \tag{3}$$

$$\mathbb{E}_{(q,s)}[U(w(q,s),e^*)|e^*] \geq \mathbb{E}_{(q,s)}[U(w(q,s),e)|e] \quad \forall e. \tag{4}$$

If the program has no solution, we take the cost of implementing e^* to be $+\infty$.

When the principal does not use the signal s, her cost of implementing e^* is

$$\mathcal{C}^{ns}(e^*) \equiv \min_{w(q)} \mathbb{E}_q[w(q)|e^*] \tag{5}$$

subject to

$$\mathbb{E}_q[U(w(q),e^*)|e^*] \geq \bar{U}, \tag{6}$$

$$\mathbb{E}_q[U(w(q),e^*)|e^*] \geq \mathbb{E}_q[U(w(q),e)|e] \quad \forall e. \tag{7}$$

[4]It is well known that when the action space is finite (as we will assume throughout the paper), there is no loss of generality in assuming that it lies on the real line. Therefore, as long as we retain the assumption of a finite action space, our model can accommodate multidimensional actions. Since it is typically impossible to order multidimensional actions, it is important to obtain results that hold beyond the FOA.

[5]Holmstrom (1979) avoids this issue by assuming that either the signal is informative for all effort levels or for no effort level.

Let $w^*(q)$ be a solution of Program (5)-(7). Since $w(q,s) = w^*(q)$ satisfies the constraints of Program (2)-(4) and costs $C^{ns}(e^*)$, it follows that $\mathcal{C}^s(e^*) \leq \mathcal{C}^{ns}(e^*)$: a signal cannot have negative value. The signal has *positive value* for implementing e^* if $\mathcal{C}^s(e^*) < \mathcal{C}^{ns}(e^*)$ – i.e. the cost of doing so is strictly lower when the contract is contingent on the signal – and *zero value* if $\mathcal{C}^s(e^*) = \mathcal{C}^{ns}(e^*)$.

We now state Holmstrom's original theorem.

Theorem. (Informativeness Principle): *Assume that the utility function is additively separable and that the FOA is valid. Suppose states are distributed according to a continuously differentiable probability density function $f(q,s|e)$. The signal has zero value for implementing e^* if and only if there exists a function ϕ for which*

$$\frac{f_e(q,s|e^*)}{f(q,s|e^*)} = \phi(q,e^*) \tag{8}$$

for almost all q, s.

The left-hand side of (8) corresponds to the change in the likelihood ratio $\frac{f(q,s|e^*+\Delta e)}{f(q,s|e^*)}$ for infinitesimal changes in effort $\Delta e \approx 0$. Since only the local IC matters when the FOA is valid, a signal has positive value for implementing e^* if and only if it affects the local likelihood ratio – i.e. it is informative about whether the agent has deviated to an adjacent effort level. However, if the FOA is invalid, the agent may be tempted to deviate to a non-adjacent effort level. Thus, even if a signal affects the local likelihood ratio, it may have no value. In Example 1, we show that when effort has "stochastic increasing returns to scale", non-local incentive constraints bind, and so the informativeness principle may not hold.

Example 1. *The agent has additively separable utility, and we normalize effort to be measured in cost units: $K(e) = \bar{K}$ and $G(e) = -e$.*[6] *The effort space is the unit interval: $\mathcal{E} = [0,1]$. Suppose the principal wishes to implement effort $e^* = 1$.*

Conditional on effort e, states are distributed according to the probability density function $f(q,s|e)$. Let $\bar{f}(q|e) = \int f(q,s|e)ds$ denote the marginal distribution of output and $\bar{F}(q|e)$ denote the associated cumulative distribution function ("CDF"). Suppose that $f(q,s|e=0)$ and $f(q,s|e=1)$ are both independent of s.

In Supplementary Appendix B.1, we show that the ICs regarding intermediate effort levels ($e \in (0,1)$) do not bind if $\frac{\bar{f}(q|e=1)}{\bar{f}(q|e=0)}$ is non-decreasing (monotone likelihood ratio

[6] As long as costs are increasing in effort, there is no loss of generality in assuming that costs are measured in units of effort. In this case, any non-linearity in effort costs is incorporated in the effect of effort on the probability distribution.

property, "MLRP") and $\bar{F}(q|e)$ *is concave in* e *for each* q. *Then, the only binding constraint involves the global deviation from* $e = 1$ *to* $e = 0$, *so the relevant likelihood ratio is* $\frac{f(q,s|e=1)}{f(q,s|e=0)}$, *which is not a function of* s. *Therefore, a signal* s *may affect the local likelihood ratio* $\frac{f_e(q,e|\hat{e})}{f(q,e|\hat{e})}$ *for almost all* $\hat{e}$ *(including* $e^* = 1$*) and still have zero value.*

Example 1 builds on Rogerson (1985), who shows that if the distribution satisfies the MLRP and the CDF is *convex*, then only the local ICs bind, justifying the FOA. MLRP is a standard condition that is satisfied by many standard distributions. As Rogerson argues, convexity of the CDF can be interpreted as "stochastic decreasing returns to scale": as effort increases, the probability of observing an outcome below q decreases at a decreasing rate. Our concavity condition is the opposite case, where the probability of observing an outcome below q decreases at an *increasing* rate, and can be interpreted as "stochastic increasing returns to scale." This example shows that global concavity of the CDF (in conjunction with MLRP) is sufficient to justify the use of binary effort models.

Although MLRP is a standard assumption in moral hazard models, neither global convexity (a sufficient condition for the FOA) nor global concavity (a sufficient condition for the binding IC to be associated with a boundary effort level) are likely to hold in practice. For example, as noted by Jewitt (1988), if output can be written $q = e + \epsilon$ for some random variable ϵ with density f, the CDF of q is convex (concave) if and only if f is an increasing (decreasing) function. Most standard distributions are neither everywhere increasing nor everywhere decreasing, so both global convexity and global concavity are unlikely. Thus, we do not know ex ante what ICs will bind, which motivates the generalized informativeness principle to which we now turn.

3 The Generalized Informativeness Principle

Following Grossman and Hart (1983), we assume that there are finitely many states and effort levels: $\mathcal{E} \equiv \{1, \ldots, E\}$, $\mathcal{X} \equiv \{q_1, ..., q_X\}$, and $\mathcal{S} \equiv \{1, ..., S\}$. Finite effort levels allow us to use Kuhn-Tucker methods to obtain necessary optimality conditions. With a continuum of effort levels, there is no general method for solving moral hazard problems without the FOA.[7] The probability of observing state (q, s) conditional on

[7]Since Holmstrom (1979) assumes the FOA, he is able to consider a continuum of efforts while retaining tractability, because the FOA means that only the local incentive constraint is relevant. In contrast, our model does not assume the FOA and so considers a finite action space.

effort e is denoted $p^e_{q,s}$, which we assume to be strictly positive to ensure existence of an optimal contract ("full support"). Let $h \equiv V^{-1}$ denote the inverse of the utility of money. Since V is increasing and strictly concave, h is increasing and strictly convex. Defining $u_{q,s} \equiv V\left(w_{q,s}\right)$, the principal's program can be written in terms of "utils":

$$\min_{\{u_{q,s}\}} \sum_{q,s} p^{e^*}_{q,s} h\left(u_{q,s}\right) \tag{9}$$

subject to

$$G\left(e^*\right) + K\left(e^*\right) \sum_{q,s} p^{e^*}_{q,s} u_{q,s} \geq \bar{U}, \tag{10}$$

$$\sum_{q,s} \left(K\left(e^*\right) p^{e^*}_{q,s} - K\left(e\right) p^{e}_{q,s}\right) u_{q,s} \geq G\left(e\right) - G\left(e^*\right) \quad \forall\, e \in \mathcal{E}, \tag{11}$$

where (10) and (11) are the IR and IC.

When the FOA cannot be applied, it seems that the informativeness principle simply needs to be extended to consider non-local deviations. Since we do not know what effort level the agent will deviate to, being informative about every possible deviation would appear to be a sufficient condition for a signal to have positive value. We thus define the generalized informativeness principle as follows:

Definition 1. *Let e^* be an effort to be implemented and consider a distribution p over states (q,s) such that, for all $e \neq e^*$, there exist s_e, s'_e, q_e with $\frac{p^e_{q_e,s_e}}{p^{e^*}_{q_e,s_e}} \neq \frac{p^e_{q_e,s'_e}}{p^{e^*}_{q_e,s'_e}}$. The generalized informativeness principle holds for (p, e^*) if the signal s has positive value in implementing e^*.*

We now verify whether the generalized informativeness principle actually holds. We first show that, surprisingly, it may fail – even a signal that is informative about every possible deviation (i.e. affects the likelihood ratio between e^* and every other effort level) may have zero value.

We first note that a signal can only have value when there are agency costs. Let $\bar{w}_e$ denote the wage that gives the agent his reservation utility if he exerts effort e:

$$\bar{w}_e = h\left(\frac{\bar{U} - G\left(e\right)}{K\left(e\right)}\right).$$

The principal can implement effort e^* with no agency costs if, when she offers the constant wage $\bar{w}_{e^*}$ that satisfies the IR with equality, all ICs are satisfied:

$$U(\bar{w}_{e^*}, e^*) \geq U(\bar{w}_{e^*}, e) \;\; \forall\; e. \tag{12}$$

We say that the first best is feasible for e^* if condition (12) holds. Then the principal uses a constant wage and so signals automatically have zero value. When utility is either additively or multiplicatively separable, the first best is only feasible for the least costly effort. With non-separable utility, however, it may be feasible for several different effort levels (Grossman and Hart (1983)). (The first best is never achieved in Holmstrom (1979) because he assumes additively separable utility and an interior e.)

There are three cases to consider. If no IC binds,[8] the first best is feasible and so the generalized informativeness principle automatically fails. Lemma 1, proven in Supplementary Appendix B.1, states that it holds whenever exactly one IC binds (as is the case in Holmstrom's (1979) original theorem):

Lemma 1. *Fix an effort e^* for which the first best is not feasible and a distribution p. If one IC binds, the generalized informativeness principle holds for (p, e^*).*

The third case to consider is when multiple ICs bind. When there are at least three states, it is not unusual for multiple ICs to bind. Formally, we show in Supplementary Appendix B.2 that multiple ICs bind for a non-empty and open set of parameter values. Since any non-trivial model with informative signals requires at least three states (at least two outputs and at least two signals conditional on at least one output), it is important to study the case of multiple binding ICs.

We start with an example showing that, if multiple ICs bind, the generalized informativeness principle may not hold. Our example follows Holmstrom (1979) and the subsequent literature in assuming additive separability:

Example 2. *There are three effort levels, two outputs, and two signals: $\mathcal{E} = \{1, 2, 3\}$, $\mathcal{X} = \{0, 1\}$, and $\mathcal{S} = \{0, 1\}$. Let $K(1) = K(2) = K(3) = 1$, $G(1) = G(2) = 0$, $G(3) = -1$, and $\overline{U} = 0$. Thus, $e = 1$ and $e = 2$ both cost zero and $e = 3$ costs one.*

Conditional on $e = 3$, states are uniformly distributed: $p^3_{q,s} = \frac{1}{4}\ \forall\ q, s$. For $e \in \{1, 2\}$, the conditional probabilities are:

$$p^1_{1,0} = p^2_{1,1} = \frac{1}{4}, \quad p^1_{1,1} = p^2_{1,0} = \frac{1}{8}, \quad p^1_{0,0} = p^1_{0,1} = p^2_{0,0} = p^2_{0,1} = \frac{5}{16}.$$

Note that the likelihood ratios between any two effort levels are not constant:

$$\frac{p^3_{1,1}}{p^2_{1,1}} = 1 \neq 2 = \frac{p^3_{1,0}}{p^2_{1,0}}, \quad \frac{p^3_{1,1}}{p^1_{1,1}} = 2 \neq 1 = \frac{p^3_{1,0}}{p^1_{1,0}}, \quad \frac{p^2_{1,1}}{p^1_{1,1}} = 2 \neq \frac{1}{2} = \frac{p^2_{1,0}}{p^1_{1,0}}.$$

[8] We say that a constraint is binding if removing it allows the principal to obtain a higher payoff.

Let $e = 3$ be the effort to be implemented. The principal's program is

$$\min_{\{u_{q,s}\}} h(u_{1,0}) + h(u_{1,1}) + h(u_{0,0}) + h(u_{0,1})$$

subject to the IR and the two ICs, which can be rewritten as:

$$u_{1,0} + u_{1,1} + u_{0,0} + u_{0,1} \geq 4 \tag{13}$$

$$2u_{1,1} - (u_{0,0} + u_{0,1}) \geq 16 \tag{14}$$

$$2u_{1,0} - (u_{0,0} + u_{0,1}) \geq 16. \tag{15}$$

Even though the likelihood ratios between any two effort levels are not constant, the signal has zero value: $u_{q,0} = u_{q,1}$ for $q \in \{0,1\}$. To see this, notice that when $u_{0,0} \neq u_{0,1}$, replacing both of them by $\frac{u_{0,0}+u_{0,1}}{2}$ keeps all constraints unchanged and reduces the principal's cost (because h is convex). Similarly, if $u_{1,0} \neq u_{1,1}$, replacing both of them by their average $\frac{u_{1,0}+u_{1,1}}{2}$ preserves IC and IR while reducing the principal's cost.

The intuition for the failure of the generalized informativeness principle is as follows. For $e = 2$, the likelihood ratio at state $(1,0)$ is twice as large as at $(1,1)$. To relax the second IC (15), we should increase $u_{1,0}$ and decrease $u_{1,1}$. For $e = 1$, the likelihood ratio at state $(1,1)$ is twice as large as at $(1,0)$. To relax the first IC (14), we should increase $u_{1,1}$ and decrease $u_{1,0}$. Since both the likelihood ratios $\frac{p^3_{1,0}}{p^2_{1,0}}$ and $\frac{p^3_{1,1}}{p^1_{1,1}}$ and the costs of effort levels 1 and 3 coincide, the shadow prices of both ICs are the same. Thus, the benefit from relaxing one IC exactly equals the cost from tightening the other one. As a result, it is optimal for the agent's utility not to depend on the signal.

This result requires the shadow prices of the binding ICs to exactly coincide. If we perturb either the probabilities or the utility function slightly, the benefit from relaxing each constraint will differ. We can then improve the contract by increasing utility in the state with the highest likelihood ratio under the effort associated with the IC with the highest shadow cost. This intuition suggests that counterexamples such as the one in Example 2 are non-generic. We now prove that this is indeed the case.

To establish results that can be applied to settings with additive and multiplicative separability, we hold either K or G fixed in our economy parametrization. Therefore, we refer to an economy as either a vector of parameters $\{K(e), p^e_{s,q}\}_{s,q,e}$ (which holds $G(e)$ fixed), or a vector of parameters $\{G(e), p^e_{s,q}\}_{s,q,e}$ (which holds $K(e)$ fixed). Our results still hold if we parametrize an economy by K, G, and p. However, in this case, economies with additive or multiplicative separability are non-generic.

Theorem 1, proven in Appendix A, is the main result of our paper. It states that the generalized informativeness principle generically holds: signals that are informative about deviations to all effort levels have positive value for implementing e^*.

Theorem 1. (Generalized Informativeness Principle) *Fix an effort e^* for which the first best is not feasible. For all economies except for a set of Lebesgue measure zero, the generalized informativeness principle holds.*

Theorem 1 gives a sufficient conditions for a signal to add value generically. Proposition 1 now shows that Theorem 1 contains the weakest sufficient condition possible, unless one imposes additional restrictions on the utility function. Thus, the generalized informativeness principle is a necessary condition for a signal to add value without restricting the utility function. Formally, whenever the likelihood ratios between two (possibly non-adjacent) effort levels is non-constant, there exists a utility function for which the signal has positive value:

Proposition 1. *Let $\frac{p^e_{q,s}}{p^{e^*}_{q,s}} \neq \frac{p^e_{q,s'}}{p^{e^*}_{q,s'}}$ for some q, s, s', and $e \neq e^*$. Then, there exist G and K such that s has positive value in implementing e^*.*

The proof of Proposition 1 is constructive and uses additively separable utility functions. Therefore, the result holds for any class of preferences that includes additively separable utility functions, as long as we do not restrict the cost of effort.

Finally, Holmstrom's (1979) informativeness principle is an "if and only if" result. The less surprising part shows that uninformative signals have zero value ("necessity"). The more interesting part shows that every informative signal has positive value ("sufficiency"). Our main contribution is to generalize the sufficiency part. We end by generalizing the necessity part – that uninformative signals have zero value – to settings in which the FOA is not valid and utility is not additively separable. The proof, in Supplementary Appendix B.1, holds for both discrete and continuous outputs and effort levels.[9]

Proposition 2. *Let (q, s) be either continuously or discretely distributed, and let $f(q, s|e)$ denote either the probability density function or the probability mass function. Suppose $\frac{f(q,s|e)}{f(q,s|e^*)} = \phi_{e^*}(q, e)$ for all e and almost all (q, s) under e^*. Then, the signal has zero value in implementing e^*.*

[9] As in Holmstrom (1979), Proposition 2 refers to "almost all" (q, e) since a signal will not have value unless it affects the likelihood ratio over a set of outputs that occurs with positive measure.

4 Conclusion

This paper generalizes the original Holmstrom (1979) informativeness principle to settings in which the first-order approach cannot be applied. In such settings, even if a signal affects the local likelihood ratio, it may have no value for the contract if it is global deviations that are relevant. It seems natural to think that a stronger condition will be sufficient for a signal to have value even when there are more than two possible efforts and the FOA cannot be applied: that it affects all likelihood ratios, not just ones involving adjacent effort levels. However, we show that even this generalized informativeness principle will not hold if multiple incentive constraints bind with the same shadow price. While the case in which multiple constraints bind is not knife-edge, the case in which they have the same shadow price is, and so the generalized informativeness principle holds generically. In sum, the paper provides a condition, stronger than in the original formulation of the informativeness principle, that is generically sufficient for a signal to have value even when the first-order approach cannot be applied. Moreover, this condition is necessary for a signal to have value without restricting the utility function.

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A Proofs

A.1 Proof of Theorem 1

The proof will use the following corollary of Sard's Theorem:

Corollary 1. *Let $X \subset \mathbb{R}^n$ and $\Xi \subset \mathbb{R}^p$ be open, $F : X \times \Xi \to \mathbb{R}^m$ be continuously differentiable, and let $n < m$. Suppose that for all (q, θ) such that $F(q, \theta) = 0$, $DF(q, \theta)$ has rank m. Then, for all θ except for a set of Lebesgue measure zero, $F(q, \theta) = 0$ has no solution.*

For simplicity, suppose that only two ICs bind; it is straightforward but notationally cumbersome to generalize the analysis for more than two binding ICs. Without loss of generality (renumbering effort levels if necessary), let $e^* = 3$ denote the implemented effort, and let $e = 1$ and $e = 2$ denote the two effort levels with binding ICs. By assumption, the first best is not feasible for $e^* = 3$. The principal's program is

$$\min_{u_{q,s}} \sum_{q=q_1}^{q_X} \sum_{s=1}^{S} p_{q,s}^{e^*} h\left(u_{q,s}\right) \tag{16}$$

$$\text{subject to} \qquad G\left(e^*\right) + K\left(e^*\right) \sum_{q=q_1}^{q_X} \sum_{s=1}^{S} p_{q,s}^{e^*} u_{q,s} \geq \bar{U}, \tag{17}$$

$$G\left(e^*\right) + K\left(e^*\right) \sum_{q=q_1}^{q_X} \sum_{s=1}^{S} p_{q,s}^{e^*} u_{q,s} \geq G\left(e\right) + K\left(e\right) \sum_{q=q_1}^{q_X} \sum_{s=1}^{S} p_{q,s}^{e} u_{q,s} \ \ \forall\ e. \tag{18}$$

Following the parametrization of an economy, we keep either $\mathbf{G} \equiv (G(3), G(2), G(1))$ or $\mathbf{K} \equiv (K(3), K(2), K(1))$ constant (where bold letters denote vectors). Accordingly, we introduce the vector Θ, where either $\Theta = \mathbf{K}$ (if $\mathbf{G}$ is being held constant) or $\Theta = \mathbf{G}$ (if $\mathbf{K}$ is being held constant). Here, we consider the case in which the IR (17) binds. The case where it does not bind is analogous and presented in Supplementary Appendix B.1.

The (necessary) first-order condition with respect to $u_{q,s}$ is

$$-p_{q,s}^{e^*} h'\left(u_{q,s}\right) - \mu_1 K(1) p_{q,s}^{1} - \mu_2 K(2) p_{q,s}^{2} + \lambda K\left(e^*\right) p_{q,s}^{e^*} = 0 \ \ \forall\ q, s, \tag{19}$$

where μ_1 and μ_2 are the Lagrange multipliers on the ICs for deviations to $e = 1$ and $e = 2$, respectively, and λ is the Lagrange multiplier on the IR.

For the agent's payments to be independent of the signal, the system of equations (17), (18), and (19) must have as a solution $u_{q,s} = u_q \; \forall\, q, s$. Combining these equations, they can be written as $F(\mathbf{u}, \mu_1, \mu_2, \lambda; \Theta, \mathbf{p}) = 0$, where

$$F\Bigg(\underbrace{\mathbf{u}}_{X}, \underbrace{\mu_1, \mu_2, \lambda}_{3}; \underbrace{\Theta}_{3}, \underbrace{\mathbf{p}}_{3XS}\Bigg) \equiv \left[\begin{array}{c} p^3_{1,1} h'(u_1) + \mu_1 K(1) p^1_{1,1} + \mu_2 K(2) p^2_{1,1} - \lambda K(3) p^3_{1,1} \\ \vdots \\ p^3_{1,S} h'(u_1) + \mu_1 K(1) p^1_{1,S} + \mu_2 K(2) p^2_{1,S} - \lambda K(3) p^3_{1,S} \\ \vdots \\ p^3_{X,1} h'(u_X) + \mu_1 K(1) p^1_{X,1} + \mu_2 K(2) p^2_{X,1} - \lambda K(3) p^3_{X,1} \\ \vdots \\ p^3_{X,S} h'(u_X) + \mu_1 K(1) p^1_{X,S} + \mu_2 K(2) p^2_{X,S} - \lambda K(3) p^3_{X,S} \\ \sum_{q=1}^{X} u_q K(3) \sum_s p^3_{q,s} + G(3) - \bar{U} \\ \sum_{q=1}^{X} u_q K(2) \sum_s p^2_{q,s} + G(2) - \bar{U} \\ \sum_{q=1}^{X} u_q K(1) \sum_s p^1_{q,s} + G(1) - \bar{U} \end{array}\right].$$

The rest of the proof verifies that the derivative of F has full row rank so we can apply Corollary 1. We write this derivative as

$$DF = \left[\begin{array}{cccccc} A_{XS\times X} & C_{XS\times 3} & D_\Theta & H^3_{XS\times XS} & H^2_{XS\times XS} & H^1_{XS\times XS} \\ B_{3\times X} & \mathbf{0_{3\times 3}} & E_\Theta & J^3_{3\times XS} & J^2_{3\times XS} & J^1_{3\times XS} \end{array}\right].$$

Matrices $A_{XS\times X}$ and $B_{3\times X}$ are, respectively, the derivative of the first XS equations and the last three equations (IR and ICs) with respect to $\mathbf{u}$:

$$\begin{aligned} A_{XS\times X} &= \left[\begin{array}{cccc} h''(u_1)\mathbf{P}^3_1 & 0 & ... & 0 \\ 0 & h''(u_2)\mathbf{P}^3_2 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & h''(u_X)\mathbf{P}^3_X \end{array}\right] \\ B_{3\times X} &= \left[\begin{array}{cccc} K(3)\mathbf{P}^3_1 \cdot \mathbf{1}_S & K(3)\mathbf{P}^3_2 \cdot \mathbf{1}_S & ... & K(3)\mathbf{P}^3_X \cdot \mathbf{1}_S \\ K(2)\mathbf{P}^2_1 \cdot \mathbf{1}_S & K(2)\mathbf{P}^2_2 \cdot \mathbf{1}_S & ... & K(2)\mathbf{P}^2_S \cdot \mathbf{1}_S \\ K(1)\mathbf{P}^1_1 \cdot \mathbf{1}_S & K(1)\mathbf{P}^1_2 \cdot \mathbf{1}_S & ... & K(1)\mathbf{P}^1_S \cdot \mathbf{1}_S \end{array}\right], \end{aligned}$$

where $\mathbf{P}^e_q = \left(p^e_{q,1}, ..., p^e_{q,S}\right)'$ and $\mathbf{1}_S \equiv (1, 1, ..., 1)$ is the vector of ones with length S. The derivative of the first XS and last three equations with respect to the multipliers

μ_1, μ_2, and λ are, respectively,

$$C_{XS\times 3} = \begin{bmatrix} K(1)p^1_{1,1} & K(2)p^2_{1,1} & -K(3)p^3_{1,1} \\ \vdots & & \\ K(1)p^1_{1,S} & K(2)p^2_{1,S} & -K(3)p^3_{1,S} \\ \vdots & & \\ K(1)p^1_{X,1} & K(2)p^2_{X,1} & -K(3)p^3_{X,1} \\ \vdots & & \\ K(1)p^1_{X,S} & K(2)p^2_{X,S} & -K(3)p^3_{X,S} \end{bmatrix} \tag{20}$$

and the null matrix $\mathbf{0}_{3\times 3}$. The derivative of the first XS and last three equations with respect to $\{G(3), G(2), G(1)\}$ are, respectively, $\mathbf{0}_{XS\times 3}$ and the identity matrix $\mathbf{I}_3$. Thus, if $\mathbf{K}$ is constant, $\Theta = \mathbf{G}$, and we have $D_\Theta = D_{\mathbf{G}} = \mathbf{0}_{XS\times 3}$, and $E_\Theta = E_{\mathbf{G}} = \mathbf{I}_3$.

The derivatives with respect to $\{K(3), K(2), K(1)\}$ are, respectively:

$$\begin{aligned} D_{\mathbf{K}} &= \begin{bmatrix} -\lambda p^3_{1,1} & \mu_2 p^2_{1,1} & \mu_1 p^1_{1,1} \\ \vdots & & \\ -\lambda p^3_{1,S} & \mu_2 p^2_{1,S} & \mu_1 p^1_{1,S} \\ \vdots & & \\ -\lambda p^3_{X,1} & \mu_2 p^2_{X,1} & \mu_1 p^1_{X,1} \\ \vdots & & \\ -\lambda p^3_{X,S} & \mu_2 p^2_{X,S} & \mu_1 p^1_{X,S} \end{bmatrix} \\ E_{\mathbf{K}} &= \begin{bmatrix} \sum_{q=1}^X u_q \sum_s p^3_{q,s} & 0 & 0 \\ 0 & \sum_{q=1}^X u_q \sum_s p^2_{q,s} & 0 \\ 0 & 0 & \sum_{q=1}^X u_q \sum_s p^1_{q,s} \end{bmatrix} \end{aligned}$$

Thus, if $\mathbf{G}$ is constant, $\Theta = \mathbf{K}$, and we have $D_\Theta = D_{\mathbf{K}}$, and $E_\Theta = E_{\mathbf{K}}$.

The derivatives with respect to $\left(p^3_{q,s}\right)$ are:

$$H^3_{XS\times XS} = \begin{bmatrix} [h'(u_1) - K(3)\lambda]\,\mathbf{I}_S & \mathbf{0}_{S\times S} & ... & \mathbf{0}_{S\times S} \\ \mathbf{0}_{S\times S} & [h'(u_2) - K(3)\lambda]\,\mathbf{I}_S & ... & \mathbf{0}_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{S\times S} & \mathbf{0}_{S\times S} & ... & [h'(u_X) - K(3)\lambda]\,\mathbf{I}_S \end{bmatrix}$$

and

$$J^3_{3\times XS} = \begin{bmatrix} u_1 K(3)\mathbf{1}_S & ... & u_X K(3)\mathbf{1}_S \\ \mathbf{0}_S & ... & \mathbf{0}_S \\ \mathbf{0}_S & ... & \mathbf{0}_S \end{bmatrix}.$$

The derivatives with respect to $(p^2_{q,s})$ and $(p^1_{q,s})$ are, respectively:

$$H^2_{XS\times XS} = \begin{bmatrix} \mu_2 K(2)\mathbf{I}_S & \mathbf{0}_{S\times S} & ... & \mathbf{0}_{S\times S} \\ \mathbf{0}_{S\times S} & \mu_2 K(2)\mathbf{I}_S & ... & \mathbf{0}_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{S\times S} & \mathbf{0}_{S\times S} & ... & \mu_2 K(2)\mathbf{I}_S \end{bmatrix} = \mu_2 K(2)\mathbf{I}_{XS}$$

$$J^2_{3\times XS} = \begin{bmatrix} \mathbf{0}_S & ... & \mathbf{0}_S \\ u_1 K(2)\mathbf{1}_S & ... & u_X K(2)\mathbf{1}_S \\ \mathbf{0}_S & ... & \mathbf{0}_S \end{bmatrix}$$

and

$$H^1_{XS\times XS} = \mu_1 K(1)\mathbf{I}_{XS}$$

$$J^1_{3\times XS} = \begin{bmatrix} \mathbf{0}_S & ... & \mathbf{0}_S \\ \mathbf{0}_S & ... & \mathbf{0}_S \\ u_1 K(1)\mathbf{1}_S & ... & u_X K(1)\mathbf{1}_S \end{bmatrix}.$$

Note that $DF_{\mathbf{P}} = \begin{bmatrix} H^3_{XS\times XS} & H^2_{XS\times XS} & H^1_{XS\times XS} \\ J^3_{3\times XS} & J^2_{3\times XS} & J^1_{3\times XS} \end{bmatrix}$ has $XS{+}3$ rows and $3XS$ columns. Since $XS{+}3 < 3XS$, it suffices to show that $DF_{\mathbf{P}}$ has full row rank: for any $\mathbf{y} \in \mathbb{R}^{XS+3}$,

$$\underbrace{\mathbf{y}}_{1\times(XS+3)} \times \underbrace{DF_{\mathbf{P}}}_{(XS+3)\times 3XS} = \underbrace{\mathbf{0}}_{1\times 3XS} \implies \mathbf{y} = \underbrace{\mathbf{0}}_{1\times(XS+3)}.$$

Let $DF_{\mathbf{P}_i} = \begin{bmatrix} H^i_{XS\times XS} \\ J^i_{3\times XS} \end{bmatrix}$. First, expanding $\mathbf{y} \times DF_{\mathbf{P}_2} = \mathbf{0}$ gives:

$$\begin{aligned} &\mu_2 K(2) y_1 + u_1 K(2) y_{XS+2} = ... = \mu_2 K(2) y_S + u_1 K(2) y_{XS+2} = 0 \\ &\mu_2 K(2) y_{S+1} + u_2 K(2) y_{XS+2} = ... = \mu_2 K(2) y_{2S} + u_2 K(2) y_{XS+2} = 0 \\ &\vdots \\ &\mu_2 K(2) y_{S(X-1)+1} + u_X K(2) y_{XS+2} = ... = \mu_2 K(2) y_{XS} + u_X K(2) y_{XS+2} = 0. \end{aligned}$$

Dividing through by $K(2) > 0$ and rearranging gives:

$$\begin{aligned} \mu_2 y_1 &= ... = \mu_2 y_S = -u_1 y_{XS+2} \\ \mu_2 y_{S+1} &= ... = \mu_2 y_{2S} = -u_2 y_{XS+2} \\ &\vdots \\ \mu_2 y_{S(X-1)+1} &= ... = \mu_2 y_{XS} = -u_X y_{XS+2}. \end{aligned} \tag{21}$$

Similarly, expanding $\mathbf{y} \times DF_{\mathbf{P}_1} = \mathbf{0}$ yields

$$\begin{aligned}
\mu_1 K(1) y_1 &= ... = \mu_1 K(1) y_S = -u_1 K(1) y_{XS+3} \\
\mu_1 K(1) y_{S+1} &= ... = \mu_1 K(1) y_{2S} = -u_2 K(1) y_{XS+3} \\
&\vdots \\
\mu_1 K(1) y_{S(X-1)+1} &= ... = \mu_1 K(1) y_{XS} = -u_X K(1) y_{XS+3}.
\end{aligned} \tag{22}$$

with $K(1) > 0$. Recall that $\mu_1 \geq 0$ and $\mu_2 \geq 0$ with at least one of them strict. Thus,

$$\begin{aligned}
y_1 &= ... = y_S =: \bar{y}^1 \\
y_{S+1} &= ... = y_{2S} =: \bar{y}^2 \\
&\vdots \\
y_{S(X-1)+1} &= ... = y_{XS} =: \bar{y}^X.
\end{aligned}$$

From equation (21), we have:

$$\begin{aligned}
&\mu_2 \bar{y}^1 = -u_1 y_{XS+2} \\
&\vdots \\
&\mu_2 \bar{y}^X = -u_X y_{XS+2}.
\end{aligned} \tag{23}$$

Second, recall that $DF_{(\mu_1,\mu_2,\lambda)} = \begin{bmatrix} C_{XS\times 3} \\ \mathbf{0}_{3\times 3} \end{bmatrix}$, where $C_{XS\times 3}$ is described in (20). Thus, $y \times DF_{(\mu_1,\mu_2,\lambda)} = \mathbf{0}$ gives

$$\sum_{q,s} \bar{y}^q K(1) p^1_{q,s} = 0, \quad \sum_{q,s} \bar{y}^q K(2) p^2_{q,s} = 0, \quad \sum_{q,s} \bar{y}^q K(3) p^3_{q,s} = 0 \quad \forall q. \tag{24}$$

Multiplying both sides of the first equation in (24) by $\mu_2 \geq 0$:

$$\mu_2 \sum_{q,s} \bar{y}^q K(1) p^1_{q,s} = K(1) \sum_{q,s} \left(\mu_2 \bar{y}^q\right) p^1_{q,s} = 0. \tag{25}$$

However, from equation (23), we have

$$K(1) \sum_{q,s} \left(\mu_2 \bar{y}^q\right) p^1_{q,s} = -y_{XS+2} K(1) \sum_{q,s} u_q p^1_{q,s} = -y_{XS+2}(\bar{U} - G(1)), \tag{26}$$

where the last equality follows from the IC for $e = 1$. Let $G(1) \neq \bar{U}$ (the set of parameters for which $\bar{U} = G(1)$ have zero Lebesgue measure). Then, (25) and (26) imply $y_{XS+2} = 0$. Applying this logic to the second equation in (24) gives $y_{XS+3} = 0$.

Third, recall from equations (21) and (22) that, $\forall\ q$,

$$\mu_2 \bar{y}^q = -u_q y_{XS+2} \text{ and } \mu_1 \bar{y}^q = -u_q y_{XS+3}.$$

Moreover, $\mu_1 \geq 0$ and $\mu_2 \geq 0$ with at least one of them strict. Since $y_{XS+2} = y_{XS+3} = 0$, we have $\mu_1 \bar{y}^q = \mu_2 \bar{y}^q = 0$. Since either $\mu_1 \neq 0$ or $\mu_2 \neq 0$, this implies $\bar{y}^q = 0\ \forall\ q$.

Fourth, expanding $\mathbf{y} \times DF_{\mathbf{P}_3} = \mathbf{0}$ gives:

$$\begin{aligned}
y_1 \left[h'(u_1) - K(3)\lambda\right] + y_{XS+1} u_1 K(3) &= 0, \\
&\vdots \\
y_S \left[h'(u_1) - K(3)\lambda\right] + y_{XS+1} u_1 K(3) &= 0, \\
y_{S+1} \left[h'(u_2) - K(3)\lambda\right] + y_{XS+1} u_2 K(3) &= 0, \\
&\vdots \\
y_{2S} \left[h'(u_2) - K(3)\lambda\right] + y_{XS+1} u_2 K(3) &= 0, \\
&\vdots \\
y_{S(X-1)+1} \left[h'(u_X) - K(3)\lambda\right] + y_{XS+1} u_X K(3) &= 0, \\
&\vdots \\
y_{XS} \left[h'(u_X) - K(3)\lambda\right] + y_{XS+1} u_X K(3) &= 0.
\end{aligned}$$

Since $y_1 = \cdots = y_{XS} = 0$ and $K(3) > 0$, this implies that either $u_1 = \cdots = u_X\ (= 0)$ or $y_{XS+1} = 0$. The former is impossible: such a contract either violates at least one IC, or satisfies all ICs. In the latter case, the constant wage (determined by the binding IR) would induce e^*, which contradicts the assumption that the first best is not feasible. Thus, $y_{XS+1} = 0$, and so $\mathbf{y} \times DF_{\mathbf{P}} = \mathbf{0} \Rightarrow \mathbf{y} = \mathbf{0}$. Hence, $DF_{\mathbf{P}}$ has full row rank.

A.2 Proof of Proposition 1

The proof is by construction. Since the goal is to show that there is a utility function for which the signal has value, it is sufficient to do so assuming an additive cost of effort. Take $K(\hat{e}) = 1$ and define $c(\hat{e}) \equiv -G(\hat{e})\ \forall\ \hat{e}$. Consider the relaxed program that only takes into account the IC between effort levels e^* and e:

$$\min_{u_{q,s}} \sum_{q,s} p_{q,s}^{e^*} h(u_{q,s}) \text{ s.t.} \tag{27}$$

$$\sum_{q,s} p_{q,s}^{e^*} u_{q,s} \geq c(e^*) + \bar{U}, \tag{28}$$

$$\sum_{q,s} \left(p_{q,s}^{e^*} - p_{q,s}^{e}\right) u_{q,s} \geq c(e^*) - c(e). \tag{29}$$

Let $u_{q,s}^*$ denote a solution to this program, which depends on the function c. Since this is formally identical to the principal's program in a standard two-effort model, $u_{q,s}^*$ is a function of s if and only if $c(e^*) > c(e)$.

Fix $c(e^*)$ and $c(e) < c(e^*)$, and let $\bar{u} \equiv \sup_{q,s}\{u_{q,s}^*\}$ and $\underline{u} \equiv \inf_{q,s}\{u_{q,s}^*\}$. Letting $c(\hat{e}) \geq c(e^*) - \bar{u} + \underline{u} \ \forall \ \hat{e} \notin \{e^*, e\}$ ensures that the omitted ICs do not bind, implying that the solution of the relaxed program $u_{q,s}^*$ also solves the principal's program.

B Supplementary Appendix: Not for Publication

B.1 Additional Proofs

Proof of Example 1

For notational simplicity, let $\pi^e_{q,s} \equiv f(q, s|e)$ denote the probability of state (q, s) conditional on effort e, $\bar{\pi}^e_q \equiv \int \pi^e_{q,s} ds$ denote the marginal probability of output q, and $\bar{\Pi}^e_q$ denote the associated cumulative distribution function ("CDF"). Suppose that $\pi^1_{q,s}$ and $\pi^0_{q,s}$ are both independent of s. As in Grossman and Hart (1983), it is convenient to write the principal's program in terms of "utils". Ignoring intermediate effort levels, the program is:

$$\min_V \int h(V(q))\,\bar{\pi}^1_q dq \text{ s.t.}$$

$$\int V(q)\,\bar{\pi}^1_q dq \geq \overline{U} \tag{30}$$

$$\int V(q)\left(\bar{\pi}^1_q - \bar{\pi}^0_q\right) dq \geq 1, \tag{31}$$

where $h = V^{-1}$.

We wish to study conditions under which the solution to this relaxed program also solves the original program – i.e. under which the following omitted ICs are satisfied:

$$\int_S \int_X V(q)\left(\pi^1_{q,s} - \pi^e_{q,s}\right) dq ds \geq 1 - e, \ \forall e.$$

Using the marginal distributions, we can rewrite these constraints as

$$\xi(e) \equiv \int_X V(q)\left(\bar{\pi}^1_q - \bar{\pi}^e_q\right) dq - (1 - e) \geq 0.$$

Note that $\xi(1) = 0$ and, by the binding IC (31), $\xi(0) = 0$. Thus, it suffices to show that ξ is concave.

Applying integration by parts to the solution of the relaxed program, we obtain

$$\int V(q)\left(\bar{\pi}^1_q - \bar{\pi}^e_q\right) dq = \int \dot{V}(q)\left(\bar{\Pi}^e_q - \bar{\Pi}^1_q\right) dq,$$

where $\bar{\Pi}$ is the CDF associated with $\bar{\pi}$. Substituting back in the definition of ξ yields

$$\xi(e) = \int \dot{V}(q)\left(\bar{\Pi}^q_e - \bar{\Pi}^q_1\right) dq + e - 1.$$

Since the likelihood ratio $\bar{\pi}_q^1/\bar{\pi}_q^0$ is non-decreasing in q, the solution of the relaxed program is monotonic: $\dot{V} \geq 0$. Then, since $\bar{\Pi}_q^e$ is a concave function of e, ξ is concave.

Proof of Lemma 1

Suppose that exactly one IC binds in Program (9)-(11) and let e^* be an effort for which the first best is not feasible. The necessary Kuhn-Tucker conditions from the principal's program yield, $\forall$ (q,s) in the support,

$$-h'\left(u_{q,s}\right) + \mu\left(K\left(e^*\right) - K(e')\frac{p_{q,s}^{e'}}{p_{q,s}^{e^*}}\right) + \lambda K\left(e^*\right) = 0, \tag{32}$$

where $\mu \geq 0$ is the multiplier associated with the binding IC. Subtracting these conditions in states (q,s) and (q,s') gives

$$h'\left(u_{q,s}\right) - h'\left(u_{q,s'}\right) = \mu K(e')\left(\frac{p_{q,s'}^{e'}}{p_{q,s'}^{e^*}} - \frac{p_{q,s}^{e'}}{p_{q,s}^{e^*}}\right). \tag{33}$$

If $\mu = 0$, then (33) implies a constant wage, which contradicts our assumption that the first best is not feasible.[10] Therefore, $\mu > 0$ and, because $K\left(e\right) > 0$ $\forall$ e, it follows from (33) and the convexity of h that $u_{q,s} \neq u_{q,s'}$ whenever $\frac{p_{q,s'}^{e'}}{p_{q,s'}^{e}} \neq \frac{p_{q,s}^{e'}}{p_{q,s}^{e}}$.

Proof of Theorem 1, non-binding IR

This appendix completes the proof of Theorem 1, by considering the case where the IR (17) does not bind. We can thus ignore the IR from the principal's program. The first-order condition with respect to $u_{q,s}$ is

$$-p_{q,s}^{e^*}h'\left(u_{q,s}\right) - \mu_1\left(K(1)p_{q,s}^1 - K\left(e^*\right)p_{q,s}^{e^*}\right) - \mu_2\left(K(2)p_{q,s}^2 - K\left(e^*\right)p_{q,s}^{e^*}\right) = 0 \;\; \forall q,s. \tag{34}$$

For the wage to be independent of the signal, the system of equations (18) and (34) must have as a solution $u_{q,s} = u_q$ $\forall$ q,s. We can write this system of equations using

[10] Since the agent's preferences over efforts are independent of income (Assumption (1iii)), effort e^* can be implemented with the minimum constant wage $\bar{w}_{e^*}$ if and only if it can be implemented with any other wage $w \geq \bar{w}_{e^*}$.

the function $F : \mathbb{R}^{X(1+3S)+5} \to \mathbb{R}^{XS+2}$, where

$$F\left(\underbrace{u_1, ..., u_X}_{X}, \underbrace{\mu_1, \mu_2}_{2}; \underbrace{\Theta}_{3}, \underbrace{p^e_{1,1}, ..., p^e_{X,S}}_{3XS}\right)$$

$$= \left[\begin{array}{c} p^3_{1,1}h'(u_1) + \mu_1(K(1)p^1_{1,1} - K(3)p^3_{1,1}) + \mu_2(K(2)p^2_{1,1} - K(3)p^3_{1,1}) \\ \vdots \\ p^3_{1,S}h'(u_1) + \mu_1(K(1)p^1_{1,S} - K(3)p^3_{1,S}) + \mu_2(K(2)p^2_{1,S} - K(3)p^3_{1,S}) \\ \vdots \\ p^3_{X,1}h'(u_X) + \mu_1(K(1)p^1_{X,1} - K(3)p^3_{X,1}) + \mu_2(K(2)p^2_{X,1} - K(3)p^3_{X,1}) \\ \vdots \\ p^3_{X,S}h'(u_X) + \mu_1(K(1)p^1_{X,S} - K(3)p^3_{X,S}) + \mu_2(K(2)p^2_{X,S} - K(3)p^3_{X,S}) \\ \sum_{q=1}^X u_q \left(K(2)\sum_s p^2_{q,s} - K(3)\sum_s p^3_{q,s}\right) + G(2) - G(3) \\ \sum_{q=1}^X u_q \left(K(1)\sum_s p^1_{q,s} - K(3)\sum_s p^3_{q,s}\right) + G(1) - G(3) \end{array}\right].$$

To apply Corollary 1, we need to show that DF has full row rank. It is given by:

$$DF = \left[\begin{array}{cccccc} A_{XS\times X} & C_{XS\times 2} & D_\Theta & H^3_{XS\times XS} & H^2_{XS\times XS} & H^1_{XS\times XS} \\ B_{2\times X} & \mathbf{0}_{2\times 2} & E_\Theta & J^3_{2\times XS} & J^2_{2\times XS} & J^1_{2\times XS} \end{array}\right].$$

Matrices $A_{XS\times X}$ and $B_{2\times X}$ are, respectively, the derivative of the first XS equations and the last 2 equations (ICs) with respect to $\mathbf{u}$:

$$A_{XS\times X} = \left[\begin{array}{cccc} h''(u_1)\mathbf{P}^3_1 & 0 & ... & 0 \\ 0 & h''(u_2)\mathbf{P}^3_2 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & h''(u_X)\mathbf{P}^3_X \end{array}\right],$$

$$B_{2\times X} = \left[\begin{array}{ccc} K(2)\mathbf{P}^2_1\cdot\mathbf{1}_S - K(3)\mathbf{P}^3_1\cdot\mathbf{1}_S & ... & K(2)\mathbf{P}^2_S\cdot\mathbf{1}_S - K(3)\mathbf{P}^3_X\cdot\mathbf{1}_S \\ K(1)\mathbf{P}^1_1\cdot\mathbf{1}_S - K(3)\mathbf{P}^3_1\cdot\mathbf{1}_S & ... & K(1)\mathbf{P}^1_S\cdot\mathbf{1}_S - K(3)\mathbf{P}^3_X\cdot\mathbf{1}_S \end{array}\right].$$

The derivatives with respect to the multipliers μ_1 and μ_2 are, respectively,

$$C_{XS\times 2} = \left[\begin{array}{cc} K(1)p^1_{1,1} - K(3)p^3_{1,1} & K(2)p^2_{1,1} - K(3)p^3_{1,1} \\ \vdots & \\ K(1)p^1_{1,S} - K(3)p^3_{1,S} & K(2)p^2_{1,S} - K(3)p^3_{1,S} \\ \vdots & \\ K(1)p^1_{X,1} - K(3)p^3_{X,1} & K(2)p^2_{X,1} - K(3)p^3_{X,1} \\ \vdots & \\ K(1)p^1_{X,S} - K(3)p^3_{X,S} & K(2)p^2_{X,S} - K(3)p^3_{X,S} \end{array}\right] \quad (35)$$

and the null matrix $\mathbf{0}_{2\times 2}$. The derivatives with respect to $\{G(3), G(2), G(1)\}$ are, respectively, $\mathbf{0}_{XS\times 3}$ and

$$E_{\mathbf{G}} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Thus, if $\mathbf{K}$ is constant, $\Theta = \mathbf{G}$, and we have $D_\Theta = D_{\mathbf{G}} = \mathbf{0}_{XS\times 3}$ and $E_\Theta = E_{\mathbf{G}}$.

The derivatives with respect to $\{K(3), K(2), K(1)\}$ are, respectively:

$$\begin{aligned} D_{\mathbf{K}} &= \begin{bmatrix} -\mu_1 p^3_{1,1} - \mu_2 p^3_{1,1} & \mu_2 p^2_{1,1} & \mu_1 p^1_{1,1} \\ \vdots & & \\ -\mu_1 p^3_{1,S} - \mu_2 p^3_{1,S} & \mu_2 p^2_{1,S} & \mu_1 p^1_{1,S} \\ \vdots & & \\ -\mu_1 p^3_{X,1} - \mu_2 p^3_{X,1} & \mu_2 p^2_{X,1} & \mu_1 p^1_{X,1} \\ \vdots & & \\ -\mu_1 p^3_{X,S} - \mu_2 p^3_{X,S} & \mu_2 p^2_{X,S} & \mu_1 p^1_{X,S} \end{bmatrix}, \\ E_{\mathbf{K}} &= \begin{bmatrix} -\sum_{q=1}^X u_q \sum_s p^3_{q,s} & \sum_{q=1}^X u_q \sum_s p^2_{q,s} & 0 \\ -\sum_{q=1}^X u_q \sum_s p^3_{q,s} & 0 & \sum_{q=1}^X u_q \sum_s p^1_{q,s} \end{bmatrix}. \end{aligned}$$

Thus, if $\mathbf{G}$ is constant, $\Theta = \mathbf{K}$, and we have $D_\Theta = D_{\mathbf{K}}$, and $E_\Theta = E_{\mathbf{K}}$.

The derivatives with respect to $\left(p^3_{q,s}\right)$, $\left(p^2_{q,s}\right)$, and $\left(p^1_{q,s}\right)$ are, respectively:

$$\begin{aligned} H^3_{XS\times XS} &= \begin{bmatrix} [h'(u_1) - K(3)(\mu_1+\mu_2)]\,\mathbf{I}_S & \mathbf{0}_{S\times S} & \dots & \mathbf{0}_{S\times S} \\ \mathbf{0}_{S\times S} & \ddots & \dots & \mathbf{0}_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{S\times S} & \mathbf{0}_{S\times S} & \dots & [h'(u_X) - K(3)(\mu_1+\mu_2)]\,\mathbf{I}_S \end{bmatrix} \\ J^3_{2\times XS} &= \begin{bmatrix} -u_1 K(3)\mathbf{1}_S & \dots & -u_X K(3)\mathbf{1}_S \\ -u_1 K(3)\mathbf{1}_S & \dots & -u_X K(3)\mathbf{1}_S \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} H^2_{XS\times XS} &= \begin{bmatrix} \mu_2 K(2)\mathbf{I}_S & \mathbf{0}_{S\times S} & \dots & \mathbf{0}_{S\times S} \\ \mathbf{0}_{S\times S} & \mu_2 K(2)\mathbf{I}_S & \dots & \mathbf{0}_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{S\times S} & \mathbf{0}_{S\times S} & \dots & \mu_2 K(2)\mathbf{I}_S \end{bmatrix} = \mu_2 \mathbf{I}_{XS}, \\ J^2_{2\times XS} &= \begin{bmatrix} u_1 K(2)\mathbf{1}_S & \dots & u_X K(2)\mathbf{1}_S \\ \mathbf{0}_S & \dots & \mathbf{0}_S \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} H^1_{XS\times XS} &= \mu_1 K(1)\mathbf{I}_{XS} \\ J^1_{2\times XS} &= \begin{bmatrix} \mathbf{0}_S & ... & \mathbf{0}_S \\ u_1 K(1)\mathbf{1}_S & ... & u_X K(1)\mathbf{1}_S \end{bmatrix}. \end{aligned}$$

Note that $DF_{\mathbf{P}} = \begin{bmatrix} H^3_{XS\times XS} & H^2_{XS\times XS} & H^1_{XS\times XS} \\ J^3_{2\times XS} & J^2_{2\times XS} & J^1_{2\times XS} \end{bmatrix}$ has $XS{+}2$ rows and $3XS$ columns. Since $XS+2 < 3XS$, it suffices to show that $DF_{\mathbf{P}}$ has full row rank to establish that DF has full row rank. We thus need to show that for any vector $\mathbf{y} \in \mathbb{R}^{XS+2}$,

$$\underbrace{\mathbf{y}}_{1\times(XS+2)} \times \underbrace{DF_{\mathbf{P}}}_{(XS+2)\times 3XS} = \underbrace{\mathbf{0}}_{1\times 3XS} \implies \mathbf{y} = \underbrace{\mathbf{0}}_{1\times(XS+2)}.$$

Let $DF_{\mathbf{P}_i} = \begin{bmatrix} H^i_{XS\times XS} \\ J^i_{2\times XS} \end{bmatrix}$. First, expanding $\mathbf{y} \times DF_{\mathbf{P}_2} = \mathbf{0}$ gives:

$$\begin{aligned} &\mu_2 K(2) y_1 + u_1 K(2) y_{XS+1} = ... = \mu_2 K(2) y_S + u_1 K(2) y_{XS+1} = 0 \\ &\mu_2 K(2) y_{S+1} + u_2 K(2) y_{XS+1} = ... = \mu_2 K(2) y_{2S} + u_2 K(2) y_{XS+1} = 0 \\ &\vdots \\ &\mu_2 K(2) y_{S(X-1)+1} + u_X K(2) y_{XS+1} = ... = \mu_2 K(2) y_{XS} + u_X K(2) y_{XS+1} = 0. \end{aligned}$$

Dividing through by $K(2) > 0$ and rearranging gives:

$$\begin{aligned} \mu_2 y_1 &= ... = \mu_2 y_S = -u_1 y_{XS+1} \\ \mu_2 y_{S+1} &= ... = \mu_2 y_{2S} = -u_2 y_{XS+1} \\ &\vdots \\ \mu_2 y_{S(X-1)+1} &= ... = \mu_2 y_{XS} = -u_X y_{XS+1}. \end{aligned} \tag{36}$$

Similarly, expanding $\mathbf{y} \times DF_{\mathbf{P}_1} = \mathbf{0}$ yields

$$\begin{aligned} \mu_1 K(1) y_1 &= ... = \mu_1 K(1) y_S = -u_1 K(1) y_{XS+2} \\ \mu_1 K(1) y_{S+1} &= ... = \mu_1 K(1) y_{2S} = -u_2 K(1) y_{XS+2} \\ &\vdots \\ \mu_1 K(1) y_{S(X-1)+1} &= ... = \mu_1 K(1) y_{XS} = -u_X K(1) y_{XS+2} \end{aligned} \tag{37}$$

with $K(1) > 0$. Recall that $\mu_1 \geq 0$ and $\mu_2 \geq 0$ and at least one of them is strict. Thus,

$$\begin{aligned} y_1 &= ... = y_S =: \bar{y}^1 \\ y_{S+1} &= ... = y_{2S} =: \bar{y}^2 \\ &\vdots \\ y_{S(X-1)+1} &= ... = y_{XS} =: \bar{y}^X. \end{aligned}$$

From equation (36), we have:

$$\begin{aligned} \mu_2 \bar{y}^1 &= -u_1 y_{XS+1} \\ &\vdots \\ \mu_2 \bar{y}^X &= -u_X y_{XS+1} \end{aligned} \tag{38}$$

Second, recall that $DF_{(\mu_1,\mu_2)} = \begin{bmatrix} C_{XS\times 2} \\ \mathbf{0}_{2\times 2} \end{bmatrix}$. Thus, $\mathbf{y} \times DF_{(\mu_1,\mu_2)} = \mathbf{0}$ gives

$$\sum_{q,s} \bar{y}^q \left[K(1)p^1_{q,s} - K(3)p^3_{q,s}\right] = 0, \quad \sum_{q,s} \bar{y}^q \left[K(2)p^2_{q,s} - K(3)p^3_{q,s}\right] = 0, \quad \forall q. \tag{39}$$

Multiplying both sides of the first equation in (39) by $\mu_2 \geq 0$:

$$\mu_2 \sum_{q,s} \bar{y}^q \left[K(1)p^1_{q,s} - K(3)p^3_{q,s}\right] = K(1) \sum_{q,s} (\mu_2 \bar{y}^q)\, p^1_{q,s} - K(3) \sum_{q,s} (\mu_2 \bar{y}^q)\, p^3_{q,s} = 0. \tag{40}$$

However, from equation (38), we have

$$\begin{aligned} & K(1) \sum_{q,s} (\mu_2 \bar{y}^q)\, p^1_{q,s} - K(3) \sum_{q,s} (\mu_2 \bar{y}^q)\, p^3_{q,s} \\ = \; & -y_{XS+1} \left[K(1) \sum_{q,s} u_q p^1_{q,s} - K(3) \sum_{q,s} u_q p^3_{q,s}\right] = -y_{XS+1}(G(3) - G(1)), \end{aligned} \tag{41}$$

where the last equality follows from the binding IC for $e = 1$. Let $G(3) \neq G(1)$ (the set of parameters for which $G(3) = G(1)$ have zero Lebesgue measure). Then, (40) and (41) imply $y_{XS+1} = 0$. Applying this logic to the second equation in (39) yields $y_{XS+2} = 0$.

Third, recall from equations (36) and (37) that, $\forall\, q$,

$$\mu_2 \bar{y}^q = -u_q y_{XS+1} \text{ and } \mu_1 \bar{y}^q = -u_q y_{XS+2}.$$

Moreover, $\mu_1 \geq 0$ and $\mu_2 \geq 0$ with at least one of them strict. Since $y_{XS+1} = y_{XS+2} = 0$, we have $\mu_1 \bar{y}^q = \mu_2 \bar{y}^q = 0$. Since either $\mu_1 \neq 0$ or $\mu_2 \neq 0$, this implies $\bar{y}^q = 0 \; \forall \; q$. Thus, $\mathbf{y} \times DF_{\mathbf{P}} = \mathbf{0} \implies \mathbf{y} = \mathbf{0}$, i.e., $DF_{\mathbf{P}}$ has full row rank.

Proof of Proposition 2

The proof follows similar steps to Holmstrom (1982) and uses a trick introduced by Grossman and Hart (1983) to rewrite the principal's program as a minimization subject to linear constraints. Let the strictly convex function $h \equiv V^{-1}$ denote the inverse utility function and let F denote the cumulative distribution function ("CDF") associated with f. The principal's program can be written in terms of "utils" as:

$$\min_{u_{q,s}} \int h\left(u_{q,s}\right) dF(q,s|e^*)$$

subject to the IR and IC:

$$\begin{aligned} G\left(e^*\right) + K\left(e^*\right) \int u_{q,s} dF(q,s|e^*) &\geq \bar{U}, \\ G\left(e^*\right) + K\left(e^*\right) \int u_{q,s} dF(q,s|e^*) &\geq G\left(e\right) + K\left(e\right) \int u_{q,s} dF(q,s|e) \; \forall e. \end{aligned}$$

We will present the discrete case here. The continuous case is analogous. Suppose that $\frac{f(q,s|e)}{f(q,s|e^*)} = \phi_{e^*}(q,e) \; \forall \; q$. Then, the IC can be written as:

$$\sum_q \left(K(e^*) - K(e)\phi_e(q)\right) \left[\sum_s f(q,s|e^*) u_{q,s}\right] \geq G(e) - G(e^*) \quad \forall e.$$

Suppose $(u_{q,s})$ satisfies IR and IC and, $\forall \; q$, substitute each entry of the vector $(u_{q,1}, ..., u_{q,S})$ by the expected value: $\bar{U}_q \equiv \sum_s f(q,s|e^*) u_{q,s}$. This new vector also satisfies IC and IR. Since h is strictly convex, the principal's payoff rises if $u_{q,s}$ is not constant in s.

B.2 Multiple Binding ICs

This appendix shows that the case in which multiple ICs simultaneously bind is not knife-edge. The problem of implementing effort e at minimum cost is:

$$\min_{\{u_{q,s}\}} \sum_{q=q_1}^{q_X} \sum_{s=1}^{S} p^e_{q,s} b\left(u_{q,s}\right)$$

subject to

$$\sum_{q=q_1}^{q_X}\sum_{s=1}^{S} p^e_{q,s} u_{q,s} - c_e \geq \bar{U}$$

$$\sum_{q=q_1}^{q_X}\sum_{s=1}^{S}\left(p^e_{q,s} - p^{\tilde{e}}_{q,s}\right) u_{q,s} \geq c_e - c_{\tilde{e}} \quad \forall \tilde{e}.$$

We study the case of three effort levels and three states. This is the simplest environment to study multiple binding ICs. With two effort levels, there is only one IC; with two states, wages are two-dimensional and, since the IR and at least one IC must bind for any effort except the least costly one, we generically can only have one binding IC.

Let $\mathcal{S} = \{1,2,3\}$ and $\mathcal{E} = \{1,2,3\}$, and take the utility function $u(c) = \sqrt{c+K}$, where $K > 0$ allows for negative wages. The inverse utility function is then

$$h(u) = u^2 - K.$$

Without loss of generality, let $e = 2$ denote the implemented effort. The program is:

$$\min_{\{u_s\}} \sum_{s=1,2,3} p^2_s u^2_s$$

subject to

$$\sum_{s=1,2,3} p^2_s u_s \geq c_2$$

$$\sum_{s=1,2,3}\left(p^2_s - p^1_s\right) u_s \geq c_2 - c_1$$

$$\sum_{s=1,2,3}\left(p^2_s - p^3_s\right) u_s \geq c_2 - c_3$$

We know that IR binds. Substituting the binding IR into the two ICs, the IR and two ICs now become:

$$\sum_{s=1,2,3} p^2_s u_s = c_2$$

$$\sum_{s=1,2,3} p^1_s u_s \leq c_1 IC1 \tag{42}$$

$$\sum_{s=1,2,3} p^3_s u_s \leq c_3 IC3 \tag{43}$$

An economy is parametrized by conditional distributions and costs: $\{p_1^e, p_2^e, c_e\}_{e=1,2,3}$ (p_3^e is given by $p_3^e = 1 - p_2^e - p_1^e$). We claim that there exists an open neighborhood of parameters in which both IC_1 and IC_3 bind. To show this, we will study the maximization program where we ignore one of the ICs. If the ignored IC is satisfied at the solution of this "relaxed program," this solution solves the principal's program. We will show that, for some open set of parameter values, IC_1 fails to hold when we ignore it and IC_3 fails to hold when we ignore it. Thus, both constraints bind.

First, consider the relaxed program where we omit IC_3. The Lagrangian is

$$L = -p_1^2 u_1^2 - p_2^2 u_2^2 - p_3^2 u_3^2 + \lambda \left(p_1^2 u_1 + p_2^2 u_2 + p_3^2 u_3 - c_2\right) + \mu \left(p_1^1 u_1 + p_2^1 u_2 + p_3^1 u_3 - c_1\right),$$

which has as first-order conditions the following linear system:

$$\begin{aligned} 2u_1 &= \lambda + \mu \frac{p_1^1}{p_1^2},\ 2u_2 = \lambda + \mu \frac{p_2^1}{p_2^2},\ 2u_3 = \lambda + \mu \frac{p_3^1}{p_3^2}, \\ p_1^2 u_1 + p_2^2 u_2 + p_3^2 u_3 &= c_2, \\ p_1^1 u_1 + p_2^1 u_2 + p_3^1 u_3 &= c_1. \end{aligned}$$

We will now combine the first three equations into one by eliminating λ. From the first equation, we have $2u_1 - \mu \frac{p_1^1}{p_1^2} = \lambda$. Substituting into the second and third and combining yields the following linear system with three equations and three unknowns:

$$\begin{bmatrix} \left(\frac{p_2^1}{p_2^2} - \frac{p_1^1}{p_1^2}\right) & \left(\frac{p_1^1}{p_1^2} - \frac{p_3^1}{p_3^2}\right) & \left(\frac{p_3^1}{p_3^2} - \frac{p_2^1}{p_2^2}\right) \\ p_3^2 & p_2^2 & p_1^2 \\ p_3^1 & p_2^1 & p_1^1 \end{bmatrix} \begin{bmatrix} u_3 \\ u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ c_2 \\ c_1 \end{bmatrix},$$

which characterizes the solution of the relaxed program where we ignore IC_3.

Similarly, the solution of the relaxed program where we ignore IC_1 is given by:

$$\begin{bmatrix} \left(\frac{p_2^3}{p_2^2} - \frac{p_1^3}{p_1^2}\right) & \left(\frac{p_1^3}{p_1^2} - \frac{p_3^3}{p_3^2}\right) & \left(\frac{p_3^3}{p_3^2} - \frac{p_2^3}{p_2^2}\right) \\ p_3^2 & p_2^2 & p_1^2 \\ p_3^3 & p_2^3 & p_1^3 \end{bmatrix} \begin{bmatrix} u_3 \\ u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix}.$$

It is easy to apply Cramer's rule to obtain a closed-form solution.

Use the following vector notation: $\mathbf{p}^e \equiv (p_1^e, p_2^e, p_3^e)$. Consider $\mathbf{p}^1 = (0.1,\ 0.28,\ 0.62)$, $\mathbf{p}^2 = (0.2,\ 0.15,\ 0.65)$, $\mathbf{p}^3 = (0.3,\ 0.1,\ 0.6)$, $c_1 = 0.75$, $c_2 = 1$, $c_3 = 0.5$.

The matrix in the relaxed program where we omit IC_3 is:

$$A_1 \equiv \begin{bmatrix} \left(\frac{p_2^1}{p_2^2} - \frac{p_1^1}{p_1^2}\right) & \left(\frac{p_1^1}{p_1^2} - \frac{p_3^1}{p_3^2}\right) & \left(\frac{p_3^1}{p_3^2} - \frac{p_2^1}{p_2^2}\right) \\ p_3^2 & p_2^2 & p_1^2 \\ p_3^1 & p_2^1 & p_1^1 \end{bmatrix} = \begin{bmatrix} 1.3667 & -0.4538 & -0.9128 \\ 0.65 & 0.15 & 0.2 \\ 0.62 & 0.28 & 0.1 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} u_3 \\ u_2 \\ u_1 \end{bmatrix} = (A_1)^{-1} \begin{bmatrix} 0 \\ c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1.0703 \\ -0.3207 \\ 1.7620 \end{bmatrix},$$

where we used the fact that

$$(A_1)^{-1} = \begin{bmatrix} 0.2499 & 1.2813 & -0.2813 \\ -0.3596 & -4.2829 & 5.2829 \\ -0.5425 & 4.0478 & -3.0478 \end{bmatrix}.$$

Since A_1 has full rank, the solution is continuous in its parameters (conditional probabilities and costs) around these parameter values. Substituting in IC_3 gives

$$p_3^3 u_3 + p_2^3 u_2 + p_1^3 u_1 - c_3 = 0.6 \times 1.0703 + 0.1 \times (-0.3207) + 0.3 \times 1.7629 - 0.5 = 0.6387 > 0.$$

Thus, IC_3 fails to hold. Since the expression $p_3^3 u_3 + p_2^3 u_2 + p_1^3 u_1 - c_3$ is a continuous function of conditional probabilities, utilities, and costs, and utility is itself a continuous function of costs and probabilities, it follows that this expression is a continuous function of probabilities and costs. Thus, for parameter values in a neighborhood of the ones considered here, it is also the case that IC_3 fails to hold.

The matrix in the relaxed program where we omit IC_1 is:

$$A_3 = \begin{bmatrix} \left(\frac{p_2^3}{p_2^2} - \frac{p_1^3}{p_1^2}\right) & \left(\frac{p_1^3}{p_1^2} - \frac{p_3^3}{p_3^2}\right) & \left(\frac{p_3^3}{p_3^2} - \frac{p_2^3}{p_2^2}\right) \\ p_3^2 & p_2^2 & p_1^2 \\ p_3^3 & p_2^3 & p_1^3 \end{bmatrix} = \begin{bmatrix} -0.8333 & 0.5769 & 0.2564 \\ 0.65 & 0.15 & 0.2 \\ 0.6 & 0.1 & 0.3 \end{bmatrix},$$

which has inverse

$$(A_3)^{-1} = \begin{bmatrix} -0.3545 & 2.0909 & -1.0909 \\ 1.0626 & 5.7273 & -4.7273 \\ 0.3545 & -6.0909 & 7.0909 \end{bmatrix}.$$

The solution of the relaxed program is then

$$\begin{bmatrix} u_3 \\ u_2 \\ u_1 \end{bmatrix} = (A_3)^{-1} \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1.5455 \\ 3.3636 \\ -2.5455 \end{bmatrix}.$$

Again, the solution is continuous in the parameters in a neighborhood of the parameters selected here. Substituting in the omitted IC gives:

$$p_3^1 u_3 + p_2^1 u_2 + p_1^1 u_1 - c_1 = 0.62 \times 1.5455 + 0.28 \times 3.3636 + 0.1 \times (-2.5455) - 0.75 = 0.8955 > 0.$$

Thus, IC_1 fails to hold. As before, by continuity, this is true for all parameter values in a neighborhood of the ones chosen here.

To summarize, for all parameter values in a neighborhood of the ones chosen here, both ICs simultaneously hold. Thus it is not true that generically only one IC binds.

References

[1] Holmstrom, Bengt (1982): "Moral hazard in teams." *Bell Journal of Economics* 13, 326–340.